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Wavelet based adaptive solution of elliptic operator equations*

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ABSTRACT

This paper is concerned with the analysis of wavelet-based adaptive algorithms for adaptive grid selection in finite element method of elliptic equations. Adaptive version of finite element method is presented which can discretize the high gradient region. The selection of suitable mesh is based on the fact that wavelet coefficients are very small in the smooth region and therefore the associated nodes can be ignored. Instead of re-meshing the domain, large number of nodes from non-critical zone is eliminated and compressed stiffness matrix is solved. In this work, B-spline wavelets based multi-scale transformation operator is used. The method has very important and highly practical consequence because it suggests how to reduce the insignificant nodes of the finite element stiffness matrix. The methodology is demonstrated with the help of very simple steady state two-dimensional heat conduction equation. The proposed method is compared with the standard method of finite element.

Keywords : Wavelet; Adaptive Grid Generation; Compression

1. Introduction

Adaptive methods, such as adaptive finite element methods, are frequently used to numerically solve elliptic equations when the solution is known to have some high gradient regions. A typical algorithm uses information gained during a given stage of the computation to produce a new mesh for the next iteration. Thus, the adaptive procedure depends on the current numerical resolution of solution. The motivation for adaptive methods is that they provide flexibility to use finer resolution near singularities of the solution and thereby improve on the approximation efficiency. Since the starting (Babuska and Miller (1987), and Babuskaand and Rheinboldt (1978)) the understanding and practical realization of adaptive refinement schemes in a finite element context has been document in numerous publications (Babuskaand and Rheinboldt (1978), Bank (1983), Bank (1985), and Borenemann (1996)).

In many interesting physical systems, the domain of solution has some high gradient region, in those regions we need fine grid and in region having smooth solution coarse grid is enough. The finite element method, the most widely used technique to solve engineering problems of such domains, uses adaptive grid techniques to refine the solution at the localized domain of interest. The currently existing finite element adaptive grid techniques may be classified as either subdivision scheme that increases the number of nodes or basis refinement techniques, which uses higher order/modified basis functions.

First the finest scale finite element solution space is projected onto the scaling and wavelet spaces resulting in the decomposition of high- and low-scale components. Repetition of such a projection results in multi-scale decomposition of the fine scale solution. In the proposed wavelet projection method, the fine scale solution can be obtained by any other numerical methods also. Subsequently the properties of the wavelet functions are exploited to eliminate the nodes from the smooth region where the wavelet coefficients will not exceed a preset tolerance. This wavelet-based multi-scale transformation hierarchically filters out the less significant part of the solution, and thus provides an effective framework for the selection of significant part of the solution. In this process, the 'big' coefficient matrix at the finest level will be calculated once for complete domain whereas the 'small' adaptively compressed coefficient matrix for a priory known localized dynamic zone of high gradient (Cohen *et al.*(2001)), which will be considerably less expensive to solve, will be used for the solution in every step of solution.

The present method removes a number of implementation headaches associated with adaptive grid techniques and is a general technique independent of domain dimension. We introduce a simple, general method with minimal mathematical framework. The basic idea behind the adaptive solution is simply based on the analysis of wavelet coefficients, which gives information about the region where sharp change is starting or ending. The resulting algorithm, while capturing full generality of method, is surprisingly simple. The method has very important and highly practical consequence because it reduces the computational time significantly.

2. B-Spline Wavelet Transform:

The concept of multiresolution analysis is to interpolate an unknown field at a coarse level by means of so-called scaling functions. Any improvement to the initial approximation consists in adding 'details' wherever required provided by new functions known as wavelets. Thus it is well suited for

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multiscale solutions. A multi-resolution analysis is a nested sequence

$$\mathrm{V}_0 \subset \mathrm{V}_1 \subset \cdots \subset \mathrm{L}^2(\mathrm{R}),$$

satisfying the following properties:

- $\mathrm{V}_j \subset \mathrm{V}_{j+1}$,
- $\mathrm{f(x)} \in \mathrm{V}_j \Leftrightarrow \mathrm{f(2x)} \in \mathrm{V}_{j+1}$,
- $\mathrm{f(x)} \in \mathrm{V}_0 \Leftrightarrow \mathrm{f(x+1)} \in \mathrm{V}_0$,
- $\overline{\cup \mathrm{V}_j}$ is dense in $L^2(R)$; $\cap V_j = \{0\}$.

Each subspace V_j is spanned by a set of scaling function $\{\varphi_{j,k}(x), \forall k \in Z\}$. The complement of V_j in V_{j+1} is defined as subspace W_j such that:

$$V_{j+1} = V_j \oplus W_j \qquad \forall j \in Z,$$

The space V_{j+1} can be decomposed in a consecutive manner as:

$$V_{j+1} = V_0 \oplus W_0 \oplus W_1 \oplus W_2 ... \oplus W_j \quad .$$

The basis functions in W_j are called wavelet functions and are denoted by $\psi_{j,k}$. These wavelet and scaling functions in different scales are used for wavelet-based multi-scaling. Approximating a function $f \in L^2(R)$ by its projection $P_j f$ onto the space V_j:

$$P_j f = \sum_{k=-\infty}^{\infty} c_k \varphi_{j,k}.$$

Let us denote the projection of f on W_j as $Q_j f$. Then we have

$$P_j f = P_{j-1} f + Q_{j-1} f.$$

In the multi-scaling, scaling coefficients $c_{j,k}$ decomposed into the scaling coefficients $c_{j-1,k}$ of the approximation $P_{j-1} f$ and wavelet coefficients $d_{j-1,k}$ of $Q_{j-1,k} f$ at the next coarser scale.

3. Mathematical Models

In two spatial dimensions, the steady state heat conduction equation:

$$-\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) = Q \qquad (1)$$

where k=thermal conductivity of a material in W/m°c, Q= internal heat generation per unit volume in(W/ m^3), $q_x = -k\frac{dT}{dx}$ and $q_y = -k\frac{dT}{dy}$ are heat flux in x and y directions. $T=T(x, y)$ is a temperature field in the medium. With boundaries given in Fig. 1 $T = T_0$ on S_t, $q_n = q_0$ on S_q, $q_n = h(T - T_\infty)$ on S_c

Boundary Conditions:

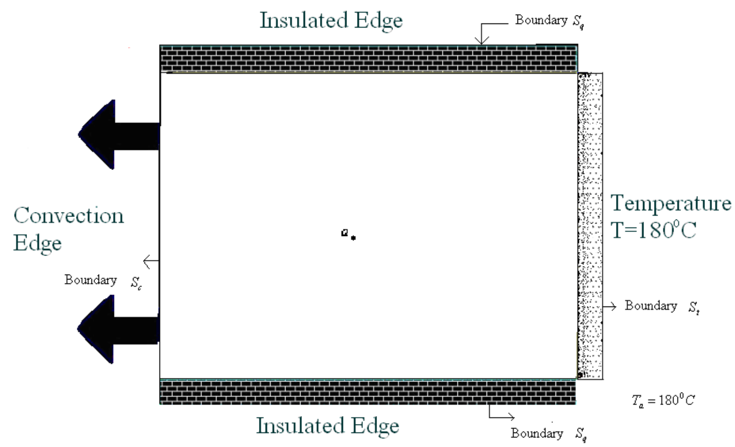


Fig.1 Boundary Condition for 2D steady state heat transfer with temperature at node a $T_a = 180°C$

The weak form of the equation (1) can be obtained as:

$$\iint w\left[-\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) - Q\right] dxdy = 0\,. \qquad (2)$$

For every w constructed from the same basis function as those used for T and satisfying w=0 on S_t.

$$\iint \left\{-\left[k\frac{\partial T}{\partial x}\frac{\partial w}{\partial x} + k\frac{\partial T}{\partial y}\frac{\partial w}{\partial y}\right] + Qw\right\}dxdy$$
$$+\iint \left[\frac{\partial}{\partial x}\left(wk\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(wk\frac{\partial T}{\partial y}\right)\right]dxdy = 0. \quad (3)$$

From the given condition $q_x = -k\frac{dT}{dx}$ and $q_y = -k\frac{dT}{dy}$, and the divergence theorem, the second integral in the above equation is

$$\iint \left[\frac{\partial}{\partial x}(wq_x) + \frac{\partial}{\partial y}(wq_y)\right]dxdy = -\int_S w[q_x n_x + q_y n_y]ds$$
$$= -\int_S w\, q_n\, ds, \qquad (4)$$

where n_x and n_y are the direction cosines of the unit normal $\boldsymbol{n}$ to the boundary and $q_n = q_x n_x + q_y n_y = q.n$ is the normal heat flow along the unit outward normal, which is specified by boundary conditions. Since $S = S_t + S_c + S_q$, w=0 on S_t, $q_n = q_0$ on S_q, and $q_n = h(T - T_\infty)$ on S_c, then equation (3) reduces to

$$0=\iint -\left[k\frac{\partial T}{\partial x}\frac{\partial w}{\partial x}+k\frac{\partial T}{\partial y}\frac{\partial w}{\partial y}-Qw\right]dxdy - \int_{S_c} w\,h(T-T_\infty)\,ds-\int_{S_q} w\,q_0 ds, \quad (5)$$

where w is the test function. Substituting interpolation function,

$$T=\sum_{j=1}^{n}T_j^e\psi_j^e(x,y) \quad (6)$$

in equation (5),leads to following finite element equation

$$\sum_{j=1}^{n}\left(K_{ij}^e+H_{ij}^e\right)T_j^e=F_j^e+P_j^e, \quad (7)$$

Where

$$\left.\begin{aligned} K_{ij}^e&=\int_e k\left(\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x}+\frac{\partial\psi_i^e}{\partial y}\frac{\partial\psi_j^e}{\partial y}\right)dxdy\\ F_i^e&=\int_e Q\,\psi_i^e dxdy-\int_{Sq}q_0^e\psi_i^e ds\\ H_{ij}^e&=h\int_{Sc}\psi_i^e\psi_j^e ds,\qquad P_i^e=h\int_{Sc}\psi_i^e T_\infty ds\end{aligned}\right\}. \quad (8)$$

The coefficients H_{ij}^e and P_i^e due to the convection boundary conditions can be computed by evaluating boundary integrals. These coefficients must be computed only for those elements and boundaries that are subjected to a convection boundary condition.

4. Wavelet Based Adaptive Method

Most adaptive strategies exploit the fact that wavelet coefficients convey detailed information on the local regularity of a function and thereby allow the detection of high gradient and low gradient regions. The rule of thumb is that whenever wavelet coefficients of the currently computed solution are large in modulus, additional refinements necessary in that region. In some sense, this amounts to using the size of the computed coefficients as local *a-posteriori* error indicators. Note that here refinement has a somewhat different meaning than in the finite element setting. There are adaptive spaces results from refining a mesh. In the wavelet context refinement means to add suitably selected further basis functions to those that are used to approximate the current solution. We refer to this as multiscaling space refinement.

The multiscaling by wavelets transformation helps to develop the compressed stiffness matrix from the finest scales solution which will be considerable less expensive to solve and also to eliminate the process of re-meshing the FE domain and re-computing the stiffness matrix. The multi-scale is performed by projecting the solution at the fine scale space V_j onto next coarser levels W_{j-1} and V_{j-1} , recursively. It is well established fact that the coefficients of wavelet space are negligible in the smooth region. Retaining grid points associated with wavelets coefficients at high gradient region and eliminating them from *a priori* known smooth region will generate adaptive grid as shown in Fig. 2. Finite element stiffness matrix at the finest scale will be transformed to the coarsest scale only once, i.e. there is no need of re-meshing and re-computing the stiffness. Compression of stiffness matrix by eliminating rows and column associated with the neglected wavelet coefficients from the smooth region will be required in all subsequent steps of adaptive solution. Fig. 3 depicts wavelet transformation.

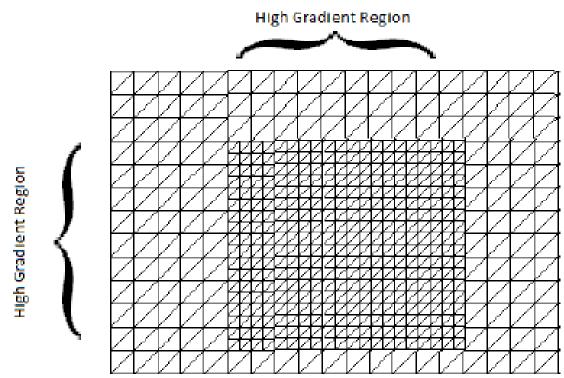


Fig. 2. Two dimensional adaptive mesh used generated

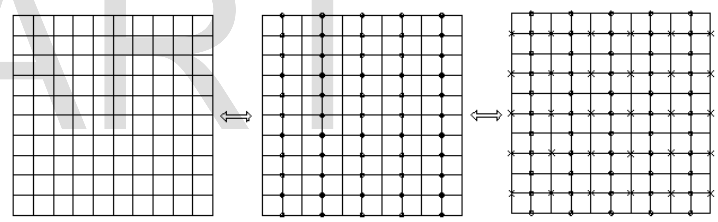


Fig 3: (a) Space V_j (b) Space $V_{j-1}\oplus W_{j-1}$ (c) Space $V_{j-2}\oplus W_{j-2}\oplus W_{j-1}$

We can retain good approximation even after discarding a large number of wavelets with small coefficients in the function, which contains isolated small part of high gradients in a large background. In other words, at any time, the computational grid should include points near the sharp gradient zone. In future, the method can be extended to solve evolution equations. Where computational grid should also consist of grid points associated with wavelets whose coefficients can possibly become significant during the next time step.

5. Procedure to select Adaptive Grid by Using Wavelets:

The finite element method with lagrange interpolation function, as a basis function is used to calculate the stiffness matrix for the whole grid. Let the finite element equation at the finest resolution J is

$$A_J T_J = f_J. \quad (9)$$

The second step B-spline wavelets given by Stollnitz at. el. [21] is applied row-wise as shown in the Fig 2 (b), the low-scale resolution of next finer scale in row-wise can be expressed by

$$W_{H,J-1}\begin{bmatrix} T_{J-1} \\ d_{J-1} \end{bmatrix} = T_J, \tag{10}$$

substitutng equation (10) in (8), we get

$$A_J W_{H,J-1}\begin{bmatrix} T_{J-1} \\ d_{J-1} \end{bmatrix} = f_J, \tag{11}$$

or $$W_{H,J-1}^T A_J W_{H,J-1}\begin{bmatrix} T_{J-1} \\ d_{J-1} \end{bmatrix} = W_{H,J-1}^T f_J. \tag{12}$$

After applying B-spline wavelets in row-wise the stiffness matrix of FEM at low resolution is obtained as:

$$A_{J-1} = W_{H,J-1}^T A_J W_{H,J-1}, \tag{13}$$

where T_{J-1} are the scaling coefficients, d_{J-1} are the wavelets coefficient.

A_{J-1} is the modified stiffness matrix after applying B-spline wavelets in row-wise.

In the next step B-spline wavelets can be applied column-wise as shown in the Fig 2(c) the next low-scale resolution can be expressed as:

$$W_{V,J-2}\begin{bmatrix} T_{J-2} \\ d_{J-2} \\ d_{J-1} \end{bmatrix} = \begin{bmatrix} T_{J-1} \\ d_{J-1} \end{bmatrix}. \tag{14}$$

Substituting (14) in (12) results in (15)

$$W_{H,J-1}^T A_J W_{H,J-1} W_{V,J-2}\begin{bmatrix} T_{J-2} \\ d_{J-2} \\ d_{J-1} \end{bmatrix} = W_{H,J-1}^T f_J. \tag{15}$$

Now, multiplying (15) on both sides by $W_{V,J-2}^T$ yields

$$W_{V,J-2}^T W_{H,J-1}^T A_J W_{H,J-1} W_{V,J-2}\begin{bmatrix} T_{J-2} \\ d_{J-2} \\ d_{J-1} \end{bmatrix} = W_{V,J-2}^T W_{H,J-1}^T f_J. \tag{16}$$

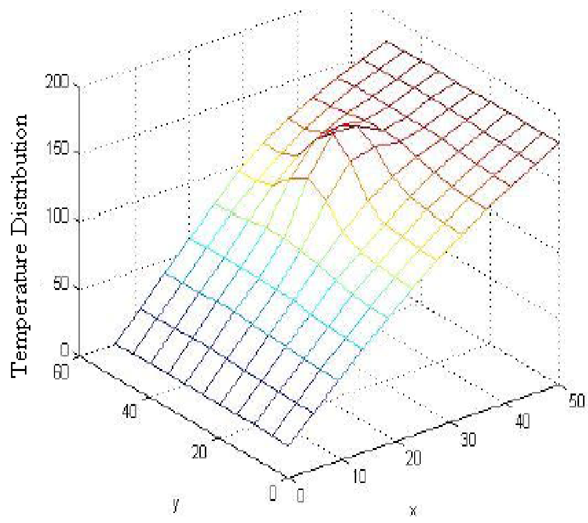
After applying B-spline wavelets in row-wise and colomn-wise alternatively the stiffness matrix obtained by FEM changes as

$$W_{V,J-2}^T W_{H,J-1}^T A_J W_{H,J-1} W_{V,J-2} = A_{J-2}. \tag{17}$$

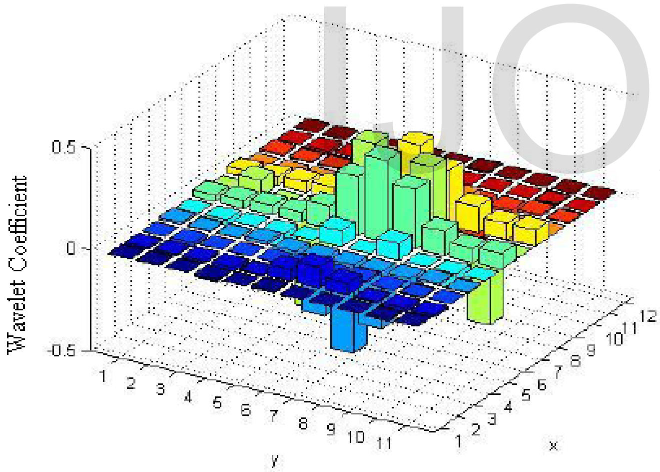
Here, $W_{H,J-1}^T$ performs row-wise wavelet transform of the equation as shown in the Fig 2(b) and $W_{V,J-2}^T$ performs column-wise wavelet transform as shown in Fig. 2(c). The alternate row and column wise wavelet transform can be continued until a desired coarsest level is achieved.

6. Results and Discussions

To illustrate the effectiveness of local behavior of the wavelets in the physical space, we consider a steady state two-dimensional heat conduction problem. In the numerical example, a rectangular plate with two opposite insulated sides (top and bottom) and a constant temperature of 180° C on the right side is considered. The other side is subjected to a convection process with T_∞ = 25°C and h=50 W/m² °C. Thermal conductivity of plate is assumed as 1.5 W/m °C. B-spline wavelet is applied on finite element stiffness matrix obtained by uniform discretization of domain by using triangular elements. To show the distribution of wavelet coefficients near high gradient and smooth region a constant temperature 180° C is applied inside the plate on one node and distribution of temperature and distribution of wavelet coefficients are shown in Figure 4(a) and 4(b) respectively. It can be observed that the wavelet coefficients are high where a sudden change in temperature 180° C is applied. The finite element stiffness matrix of 132x132 is compressed to 51x51 by eliminating small detail coefficients. To show that removal of nodes corresponding to small detail coefficients does not alter the result significantly, few typical results along x-axis and y-axis obtained from compressed matrix, are compared with those obtained by finite element method and are shown in Table 1 and Table 2. It can be observed that the results are in good agreement at every node.



(a)



(b)

Fig 4. distribution of temperature and wavelet coefficients in a rectangular plate

Table 1: Comparison of results at third row of grid from top

	Wavelet Results	FEM Results	% Error
1.	25.09294	25.09294	0
2.	39.03453	39.03454	-2.6E-05
3.	53.28598	53.28595	5.6E-05
4.	67.22758	67.22757	1.49E-05
5.	81.47899	81.47899	0
6.	95.42056	95.42064	-8.4E-05
7.	109.672	109.6721	-9.1E-05
8.	123.6136	123.6137	-8.1E-05
9.	137.8651	137.8652	-7.3E-05
10.	151.8068	151.8069	-6.6E-05
11.	166.0583	166.0583	0
12.	180	180	0

Table 2: Comparison of results at fifth column of grid from right

	Wavelet Results	FEM Results	% Error
1.	81.479	81.47895	-6.13655E-05
2.	81.47901	81.47897	-4.90924E-05
3.	81.47899	81.47899	0
4.	81.47898	81.479	2.45462E-05
5.	81.47899	81.47898	-1.22731E-05
6.	81.47897	81.47897	0
7.	81.479	81.47897	-3.68193E-05
8.	81.479	81.479	0
9.	81.47898	81.47898	0
10.	81.47897	81.47897	0
11.	81.47897	81.47897	0

7. Conclusions

In the present study we have developed a method to solve PDEs, which uses wavelet based framework for adaptive grid selection in finite element method, which show large gradient regions in the solution. In order to verify the approach, numerical solution of the 2D heat conduction equation has been performed. Using this method not only we can reduce size of stiffness matrix by elimination but also there will be no need of calculating stiffness matrix again and again in case of dynamic problem. The 2D heat conduction equation example problem and given boundary conditions is not limitation of algorithm. This choice has been made as a first step and simplicity. Due to simplicity of the problem, the advantage of grid selection may not be very visible. Nevertheless, the methodology for the grid selection is established, which can be extended to complex problem.

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