Unsteady Couette Flow of an Electrically Conducting Viscous, Incompressible Fluid with Heat Transfer in Presence of Ion-Slip

Dr.Rachna Khandelwal
Department of Mathematics,
(Jaipur Institute of Technology, Group of Institutions)
khandelwalrachna@ymail.com

Abstract

Unsteady Couette flow of an electrically conducting, viscous, incompressible fluid with heat transfer between two parallel non-conducting porous plates is investigated .A uniform suction/injection is applied perpendicular to the plate, while the fluid motion is due to a constant pressure gradient. An external uniform magnetic filed is applied perpendicular to the plates and effect of Joule and viscous dissipations are taken in account. The plates are kept at different constant temperatures. The effect of ion-slip and uniform suction and injection on both the velocity and temperature distributions are examined, discussed and shown through graphs.

Key words : Hydromagnetic , heat transfer , Hall effect , ion-slip , Couette flow , parallel plates .

1.Introduction

The magnetohydrodynamic flow between two parallel plates, known as Hartmann flow is a classical problem whose solution has many applications in magnetohydrodynamic (MHD) power generators, MHD pumps, accelerators , aerodynamic heating, electrostatic precipitation ,polymer technology , petroleum industry, purification of oil and fluid droplets and sprays etc. Hartmann and Lazarus (1937) studied the influence of a transverse uniform magnetic field on the flow of a conducting fluid between two infinite parallel, stationary and insulated plates . Then, a lot of research work concerning the Hartmann flow was conducted under different physical effects . In most of the cases, the Hall and ion-slip terms were ignored in applying Ohm's law, as they have no marked effect for small and moderate values of the magnetic field. However, the current trend for the application of magnetohydrodynamics is towards a strong magnetic field, so that the influence of electromagnetic force is noticeable by Carmer and Pai (1973). Under these conditions, the Hall current and ion-slip are important and have marked effect on the magnitude and direction of the current density and consequently on magnetic force term . Tani (1962) studied the Hall effect on the steady motion of an electrically conducting and viscous fluids in channels . Soudalgerkar et al. (1979,1986) studied the effect of the Hall currents on the steady MHD Couette flow with heat transfer. The temperature of the two plates were assumed either to be constant (1979) or to vary linearly along the plates in the direction of the flow (1986) . Abo-El-Dahab (1993) studied the effect of Hall current on the steady Hartmann flow subjected to a uniform suction and injection at the bounding plates. Attia (1998) extended the problem to the unsteady state with heat transfer, taking the Hall effect into consideration , while neglecting ion-slip,when upper plate is moving with a variable velocity $U_0(1+\varepsilon e^{i\omega t})$,while the lower plate is kept stationary. The fluid is acted upon by a constant pressure gradient, a variable suction and injection and a uniform magnetic field perpendicular to the plates. The induced magnetic field is neglected by assuming a very small magnetic Reynolds number , therefore , the uniform magnetic field B_0 is considered as the total magnetic field acting on the fluid . The plates are maintained at different but constant temperatures: T_1 and T_2 , respectively , where $T_1<T_2$. Attia (2005) studied the transient flow and heat transfer of an incompressible, viscous, electrically conducting fluid between two infinite non-conducting horizontal porous plates with the consideration of both Hall current and ion-slip. Sharma and Mishra (2002) investigated pulsatile MHD flow and heat transfer through a porous channel. Sharma and Chaturvedi (2003) presented unsteady flow and heat transfer through an electrically conducting viscous incompressible fluid between two non -conducting parallel plates under uniform transverse magnetic field . Unsteady plane Poiseuille flow and heat transfer in the presence of oscillating temperature of the lower plate have been discussed by Sharma, Chawla and Singh (2004) . Sharma and Yadav (2006) investigated pulsatile MHD flow and heat transfer through a porous channel with Ohmic effect .Sharma and Mehta (2009) presented MHD unsteady slip flow and heat transfer in a channel with slip at the permeable boundaries .

Aim of the paper is to investigate unsteady Couette flow of an electrically conducting viscous, incompressible fluid with heat transfer in presence of ion-slip.

2.Formulation of the Problem

The existence of the Hall term gives rise to a component of the velocity. Thus the velocity vector of the fluid is given by

$$\vec{v}^*(y,t)=u^*(y,t)\hat{i}+V_o(t)\hat{j}+w^*(y,t)\hat{k}. \quad \ldots(1)$$

The fluid flow is governed by the momentum equation.

$$\rho\frac{D\vec{v}^*}{Dt^*}=\mu\nabla^2\vec{v}^*-\nabla\vec{P}+\vec{J}\times\vec{B}_0, \quad \ldots(2)$$

where ρ and μ are the density and the coefficient of viscosity of the fluid respectively. If the Hall and ion- slip terms are retained, the current density J is given by

$$\vec{J} = \sigma\left[\vec{v}^* \times \vec{B}_0 - \beta(\vec{J}\times\vec{B}_0) + \frac{\beta Bi}{Bo}(\vec{J}\times\vec{B}_0)\times\vec{B}_0\right],$$

when

$$\vec{J}\times\vec{B}_0 = -\frac{\sigma B_0{}^2}{(1+BiBe)^2}\left[\{(1+BiBe)u^* + Bew^*\}\hat{i} + \{(1+BiBe)w^* - Beu^*\}\hat{k}\right],$$

σ is the electric conductivity of the fluid, β is the Hall factor, Bi is the ion-slip parameter and B_o = $|\vec{B}_o|$and Be =$\sigma\beta B_0$ is the Hall parameter.

Using equation(3) into the equation (2), we get

$$\rho\frac{\partial u^*}{\partial t^*} + \rho V_0\frac{\partial u^*}{\partial y^*} = -\frac{dp^*}{dx^*} + \mu\frac{\partial^2 u^*}{\partial y^{*2}} - \frac{\sigma B_0{}^2}{(1+BiBe)^2 + Be^2}\{1+BiBe)u^* + Bew^*\}, \quad \ldots(3)$$

$$\rho\frac{\partial w^*}{\partial t^*} + \rho V_0\frac{\partial w^*}{\partial y^*} = \mu\frac{\partial^2 w^*}{\partial y^{*2}} - \frac{\sigma B_0{}^2\{1+BiBe)w^* + Beu^*\}}{(1+BiBe)^2 + Be^2}. \quad \ldots(4)$$

The temperature distribution in the presence of rate of heat generation/absorption and Ohmic dissipation is given by

$$\rho C_p\frac{\partial T^*}{\partial t^*} + \rho C_p V_0\frac{\partial T^*}{\partial y^*} = K\frac{\partial^2 T^*}{\partial y^{*2}} + \mu\left\{\left(\frac{\partial u^*}{\partial y^*}\right)^2 + \left(\frac{\partial w^*}{\partial y^*}\right)^2\right\} + \frac{\sigma B_0{}^2(u^{*2}+w^{*2})}{(1+B_iBe)^2 + Be^2} + S^*(T^* - T_1). \quad \ldots(5)$$

3.Method of Solution

Introducing the following quantities

$$x = \frac{x^*}{h}, y = \frac{y^*}{h}, z = \frac{z^*}{h}, u = \frac{u^*}{U_0}, w = \frac{w^*}{U_0}, P = \frac{p^*}{\rho U_0^2}, t = \frac{t^* U_0}{h}, T = \frac{T^* - T_1}{T_2 - T_1}$$

Re $=\rho hU_0/\mu$, $\lambda^*=V_0/U_0$, $\lambda^* = \lambda\ (1+\varepsilon e^{i\omega t})$, Pr $=\mu C_p/\kappa$, $S=S^*h^2/C_pU_0$

$$\mathrm{Ec} = \mathrm{U_0}^2/\mathrm{C_p}\left(\mathrm{T_2} - \mathrm{T_1}\right), Ha^2 = \frac{\sigma B_0^2 h^2}{\mu}. \quad \ldots(6)$$

into the equations (3), (4) and (5),we get

$$\frac{\partial u}{\partial t} + \lambda(1+\varepsilon e^{i\omega t})\frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2} - \frac{Ha^2\{1+BiBe)u + Bew\}}{\mathrm{Re}\{(1+BiBe)^2 + Be^2\}}, \quad \ldots(7)$$

$$\frac{\partial w}{\partial t}+\lambda(1+\varepsilon e^{i\omega t})\frac{\partial w}{\partial y}=-\frac{1}{\text{Re}}\frac{\partial^2 w}{\partial y^2}-\frac{Ha^2\{1+BiBe)w+Beu\}}{\text{Re}\{(1+BiBe)^2+Be^2\}}, \quad \ldots(8)$$

$$\frac{\partial T}{\partial t}+\lambda(1+\varepsilon e^{i\omega t})\frac{\partial T}{\partial y}=-\frac{1}{\text{Re Pr}}\frac{\partial^2 T}{\partial y^2}-\frac{Ec}{\text{Re}}\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]+\frac{EcHa^2(u^2+w^2)}{\text{Re}\{(1+BiBe)^2+Be^2\}}+ST, \quad \ldots(9)$$

$$where \quad \frac{dP}{dx}=-A(1+\varepsilon e^{i\omega t}),$$

λ is the suction parameter, Re is the Reynolds number ,Ha is the Hartmann number, Pr is the Prandtl number and Ec is the Eckert number.

The initial and boundary conditions are reduced to

Initial conditions

$$u=0, w=0, T=0, \text{when} \quad t\le 0$$

The boundary conditions

$$y=-1: u=0, w=0, T=0;$$
$$y=1: u=(1+\varepsilon e^{i\omega t}), w=0, T=1. \quad \ldots(10)$$

In view of the boundary conditions, the velocity and temperature distributions are taken of the form given below

$$\left.\begin{aligned}u(y,t)&=u_o(y)+\varepsilon u_1(y)e^{i\omega t}+O(\varepsilon^2)\\ w(y,t)&=w_o(y)+\varepsilon w_1(y)e^{i\omega t}+O(\varepsilon^2)\\ T(y,t)&=T_o(y)+\varepsilon T_1(y)e^{i\omega t}+O(\varepsilon^2)\end{aligned}\right\} \quad \ldots(11)$$

When ε=0, the problem reduces to steady free convective flow which is governed by the following equations

$$\frac{1}{\text{Re}}\frac{\partial^2 u_0}{\partial y^2}-\lambda\frac{\partial u_o}{\partial y}=-A+\frac{Ha^2\{(1+BiBe)u_o+Bew_o\}}{\text{Re}((1+BiBe)^2+Be^2)}, \quad \ldots(12)$$

$$-\frac{1}{\text{Re}}\frac{\partial^2 w_o}{\partial y^2}=-\lambda\frac{\partial w_0}{\partial y}-\frac{Ha^2\{(1+BiBe)w_o+Beu_o\}}{\text{Re}((1+BiBe)^2+Be^2)}, \quad \ldots(13)$$

$$\frac{1}{\mathrm{Re}\,\mathrm{Pr}}\;\frac{\partial^2 T_o}{\partial y^2}-\lambda\frac{\partial^2 T_o}{\partial y}=-\frac{Ec}{\mathrm{Re}}\left[\left(\frac{\partial u_o}{\partial y}\right)^2+\left(\frac{\partial w_o}{\partial y}\right)^2\right]-\frac{EcHa^2}{\mathrm{Re}\{(1+BiBe)^2+Be^2\}}+ST_0. \quad \ldots(14)$$

The corresponding boundary conditions are reduced to

$$y=-1: u_0=0, w_0=0, T_0=0;$$
$$y=1: u_0=1, w_0=0, T_0=1. \quad \ldots(15)$$

When ε≠0 the governing equations are given by

$$\frac{1}{\mathrm{Re}}\frac{\partial^2 u_1}{\partial y^2}-\lambda\frac{\partial u_1}{\partial y}-i\omega u_1=-A+\lambda\frac{\partial^2 u_0}{\partial y}+\frac{Ha^2\{(1+BiBe)u_1+Bew_1\}}{\mathrm{Re}\{(1+BiBe)^2+Be^2\}}, \quad \ldots(16)$$

$$\frac{1}{\mathrm{Re}}\frac{\partial^2 w_1}{\partial y^2}-\lambda\frac{\partial w_1}{\partial y}-i\omega w_1=+\lambda\frac{\partial w_o}{\partial y}+\frac{Ha^2\{(1+BiBe)u_1+Beu_1\}}{\mathrm{Re}\{(1+BiBe)^2+Be^2\}}, \quad \ldots(17)$$

$$\frac{1}{\mathrm{Re}}\frac{\partial^2 T_1}{\partial y^2}-\lambda\frac{\partial T_1}{\partial y}-(S+i\omega)T_1=\lambda\frac{\partial To}{\partial y}-\frac{2Ec}{\mathrm{Re}}\left[\frac{\partial u_o}{\partial y}\frac{\partial u_1}{\partial y}+\frac{\partial w_o}{\partial y}\frac{\partial w_1}{\partial y}\right]+\frac{2EcHa^2(u_ou_1+w_ow_1)}{\mathrm{Re}\{(1+BiBe)^2+Be^2\}}. \quad \ldots(18)$$

The corresponding boundary conditions are reduced to

$$y=-1: u_1=0, w_1=0, T_1=0;$$
$$y=1: u_1=1, w_1=0, T_1=0. \quad \ldots(19)$$

Equations (12) to (14) and (16) to (18) are ordinary second order coupled differential equations and to be solved under the boundary conditions (15) and (19), respectively.

Since the Hartmann number Ha is small therefore velocity and temperature distributions can be expanded in the power of Ha^2 as given below

$$F=F_0+Ha^2F_1+o(Ha^4), \quad \ldots(20)$$

where F stands for any u_0, u_1, w_0, w_1, T_0 or T_1.

Using (20) into the equations (12) to (14) and (16) to (18) , and equating the like powers of Ha^2,we get

$$\frac{1}{\mathrm{Re}}\frac{d^2u_{00}}{dy^2}-\lambda\frac{du_{00}}{dy}=A \quad , \qquad ...(21)$$

$$\frac{1}{\mathrm{Re}}\frac{d^2u_{01}}{dy^2}-\lambda\frac{du_{01}}{dy}+\frac{\{(1+BiBe)u_{00}+Bew_{00}\}}{\mathrm{Re}\{(1+BiBe)^2+Be^2\}} \quad , \qquad ...(22)$$

$$\frac{1}{\mathrm{Re}}\frac{d^2u_{10}}{dy^2}-\lambda\frac{du_{10}}{dy}-i\omega u_{10}=\lambda\frac{du_{00}}{dy} \quad , \qquad ...(23)$$

$$\frac{1}{\mathrm{Re}}\frac{d^2u_{11}}{dy^2}-\lambda\frac{d^2u_{11}}{dy}-i\omega u_{11}=\lambda\frac{du_{01}}{dy}+\frac{\{(1+BiBe)u_{10}+Bew_{10}\}}{\mathrm{Re}\{(1+BiBe)\}^2} \quad , \qquad ...(24)$$

$$\frac{1}{\mathrm{Re}}\frac{d^2w_{00}}{dy^2}-\lambda\frac{dw_{00}}{dy}=0 \quad , \qquad ...(25)$$

$$\frac{1}{\mathrm{Re}}\frac{d^2w_{01}}{dy^2}-\lambda\frac{dw_{01}}{dy}=\frac{\{(1+BiBe)w_{00}+Beu_{00}\}}{\mathrm{Re}\{(1+BiBe)^2+Be^2\}} \quad , \qquad ...(26)$$

$$\frac{1}{\mathrm{Re}}\frac{d^2w_{10}}{dy^2}-\lambda\frac{dw_{10}}{dy}-i\omega w_{10}=\lambda\frac{dw_{00}}{dy} \quad , \qquad ...(27)$$

$$\frac{1}{\mathrm{Re}}\frac{d^2w_{11}}{dy^2}-\lambda\frac{dw_{11}}{dy}-i\omega w_{11}=\frac{\{(1+BiBe)w_{10}+Beu_{10}\}}{\mathrm{Re}\{(1+BiBe)^2+Be^2\}}+\lambda\frac{dw_{01}}{dy} \quad , \qquad ...(28)$$

$$\frac{1}{\mathrm{Re}\,\mathrm{Pr}}\frac{d^2T_{00}}{dy^2}-\lambda\frac{dT_{00}}{dy}+ST_{00}=-\frac{Ec}{\mathrm{Re}}\left[\left(\frac{\partial u_{00}}{\partial y}\right)+\left(\frac{\partial w_{00}}{\partial y}\right)^2\right] , \qquad ...(29)$$

$$\frac{1}{\mathrm{Re}\,\mathrm{Pr}}\frac{d^2T_{01}}{dy^2}-\lambda\frac{dT_{01}}{dy}+ST_{01}=-\frac{2Ec}{\mathrm{Re}}\left[\left(\frac{\partial u_{00}}{\partial y}.\frac{\partial u_{01}}{\partial y}+\frac{\partial w_{00}}{\partial y}.\frac{\partial w_{01}}{\partial y}\right)\right]-\frac{Ec}{\mathrm{Re}}\frac{(u_{00}{}^2+w_{00}{}^2)}{\{(1+BiBe)^2+Be^2\}} ,$$

...(30)

$$\frac{1}{\mathrm{Re}\,\mathrm{Pr}}\frac{d^2T_{10}}{dy^2}-\lambda\frac{dT_{10}}{dy}+(s-i\omega)T_{10}=-\frac{-2Ec}{\mathrm{Re}}\left[\left(\frac{du_{00}}{\partial y}.\frac{\partial u_{10}}{\partial y}+\frac{\partial w_{00}}{\partial y}.\frac{\partial w_{10}}{\partial y}\right)\right]+\lambda\frac{dT_{00}}{dy} \quad , \qquad ...(31)$$

$$\frac{1}{\operatorname{Re}\Pr}\frac{d^2T_{11}}{dy^2}-\lambda\frac{dT_{11}}{dy}+(s-i\omega)T_{11}=-\frac{2Ec}{\operatorname{Re}}\left[\frac{du_{00}}{dy}\frac{du_{11}}{dy}+\frac{du_{01}}{dy}\frac{du_{10}}{dy}+\frac{dw_{00}}{dy}\frac{dw_{10}}{dy}+\frac{dw_{01}}{dy}\frac{dw_{10}}{dy}\right]$$

$$-\frac{2Ec}{\operatorname{Re}}\frac{\{u_{00}u_{10}+w_{00}w_{10}\}}{\{(1+BeBe)^2+Be^2\}}+\frac{\lambda dT_{01}}{dy}. \quad ...(32)$$

Now, the corresponding boundary conditions are reduced to

$$\begin{aligned}&\mathrm{y}=-1: u_{00}=u_{01}=u_{10}=u_{11}=0, T_{00}=T_{01}=T_{10}=T_{11}=0, w_{00}=w_{01}=w_{10}=w_{11}=0.\\&\mathrm{y}=1: u_{00}=1=u_{01}=u_{10}, u_{11}=0, w_{00}=w_{01}=w_{10}=w_{11}=0, T_{00}=1, T_{01}=0=T_{10}=T_{11}.\end{aligned} \quad \ldots(33)$$

The equations (21) to (32) are solved under the boundary conditions (33).Through straight forward algebra, the solution of $u_{00}(y), u_{01}(y), u_{10}(y), u_{11}(y);\ w_{00}(y), w_{01}(y), w_{10}(y), w_{11}(y); T_{00}(y), T_{01}(y), T_{10}(y), and T_{11}(y)$ are known and given by

$$u_{00}(y)=A_3+A_2e^{\lambda\operatorname{Re}y}+\frac{A}{\lambda}y, \quad \ldots(34)$$

$$u_{01}(y)=A_7+A_8e^{\lambda\operatorname{Re}y}+A_9ye^{\lambda\operatorname{Re}y}+A_{10}y^2+A_{11}y, \quad \ldots(35)$$

$$u_{10}(y)=A_{18}e^{R_1y}+A_{19}e^{R_2y}+\frac{iA_2\lambda^2\operatorname{Re}}{\omega}e^{\lambda\operatorname{Re}y}+\frac{iA}{\omega}, \quad \ldots(36)$$

$$\begin{aligned}u_{11}(y)&=A_{22}e^{R_1y}+A_{23}e^{R_2y}+(iA_{24}-A_{25})e^{\lambda\operatorname{Re}y}+(iA_{26}y-A_{25})e^{\lambda\operatorname{Re}y}\\&+(iA_{26}y-A_{27})e^{\lambda\operatorname{Re}y}+(iA_{28}y+A_{29})+(A_{32}y+iA_{33})e^{RR_1y}\cos(IR_1y)\\&+(A_{34}+iA_{35})e^{RR_1y}\sin(IR_1y)+(A_{38}+iA_{39})e^{RR_2y}\cos(IR_2y)+(A_{40}+iA_{41})e^{RR_1y}Sin(IR_2y),\end{aligned} \quad \ldots(37)$$

$$w_{00}(y)=0, \quad \ldots(38)$$

$$w_{01}(y)=A_{14}+A_{15}e^{\lambda\operatorname{Re}y}+A_{16}y^2+A_{17}y, \quad \ldots(39)$$

$$w_{10}(y)=0, \quad \ldots(40)$$

$$w_{11}(y)=(A_{42}-A_{48})e^{R_1y}+(A_{43}-A_{49})e^{R_2y}+iA_{44}e^{\lambda\operatorname{Re}y}+iA_{45}y+i(\frac{\lambda}{\omega}A_{45}+A_{47})-A_{46}, \quad \ldots(41)$$

ISSN 2278-7763

$$T_{00}(y)=A_{50}e^{R_3 y}+A_{51}e^{R_4 y}-A_{52}e^{\lambda \operatorname{Re} y}-A_{53}, \qquad \ldots(42)$$

$$T_{01}(y)=A_{55}eR_3 y+A_{56}e^{R_4 y}-(A_{64}+A_{66})e^{2\lambda \operatorname{Re} y}-A_{65}ye^{2\lambda \operatorname{Re} y}$$
$$-A_{67}ye^{\lambda \operatorname{Re} y}-(A_{68}+A_{69})e^{\lambda \operatorname{Re} y}-A_{72}y^2-(A_{70}+A_{73})y-(A_{71}+A_{74}+A_{75}), \qquad \ldots(43)$$

$$T_{10}(y)=A_{76}{}^{R_5 y}+A_{77}e^{R_6 y}-A_{83}e^{(R_1+\lambda \operatorname{Re})y}-A_{84}e^{(R_1+\lambda \operatorname{Re})y}-iA_{85}e^{2\lambda \operatorname{Re} y}-A_{86}e^{R_1 y}-$$
$$A_{87}e^{R_2 y}+A_{88}e^{R_3 y}+A_{89}e^{R_4 y}-A_{90}e^{\lambda \operatorname{Re} y}, \qquad \ldots(44)$$

$$T_{11}(y)=A_{91}e^{R_5 y}+A_{92}e^{R_6 y}+(A_{93}+A_{101})e^{RR_1 y}\cos(IR_1 y)+(A_{94}+A_{102})e^{RR_1 y}\sin(IR_1 y)+(A_{95}+A_{103})e^{RR_2 y}\cos(IR_2 y)$$
$$+(A_{96}+A_{104})e^{RR_2 y}\sin(IR_2 y)+(A_{97}+A_{99})e^{2\lambda \operatorname{Re} y}+A_{103}+A_{104}+A_{105}. \qquad \ldots(45)$$

where A, A_2 to A_{105} are constants and their expressions are presented in Appendix.

4.Skin-friction

The skin-friction coefficient at the lower plate is given by

$$C_f=-\left(\frac{\partial u}{\partial y}\right)_{y=-1}=-\left[\left(\frac{\partial u_{00}}{\partial y}+Ha^2\frac{\partial u_{01}}{\partial y}\right)+\varepsilon\left(\frac{\partial u_{10}}{\partial y}+Ha^2\frac{\partial u_{11}}{\partial y}\right)e^{i\omega t}\right]_{y=-1}. \qquad \ldots(46)$$

The skin-friction coefficient at the upper plate is given by

$$C_f=-\left(\frac{\partial u}{\partial y}\right)_{y=1}=-\left[\left(\frac{\partial u_{00}}{\partial y}+Ha^2\frac{\partial u_{01}}{\partial y}\right)+\varepsilon\left(\frac{\partial u_{10}}{\partial y}+Ha^2\frac{\partial u_{11}}{\partial y}\right)e^{i\omega t}\right]_{y=1}. \qquad \ldots(47)$$

5.Rate of Heat Transfer

The rate of heat transfer in terms of the Nusselt number at the lower plate is given by

$$Nu=-\left[\left(\frac{\partial T_{00}}{\partial y}+Ha^2\frac{\partial T_{01}}{\partial y}\right)+\varepsilon\left(\frac{\partial T_{10}}{\partial y}+Ha^2\frac{\partial T_{11}}{\partial y}\right)e^{i\omega t}\right]_{y=-1} \qquad \ldots(48)$$

The rate of heat transfer in terms of the Nusselt number at the upper plate is given by

$$Nu=-\left[\left(\frac{\partial T_{00}}{\partial y}+Ha^2\frac{\partial T_{01}}{\partial y}\right)+\varepsilon\left(\frac{\partial T_{10}}{\partial y}+Ha^2\frac{\partial T_{11}}{\partial y}\right)e^{i\omega t}\right]_{y=1} \qquad \ldots(49)$$

6. Tables and Graphs

Ha	Bi	Bo	
1	1	1	I
3	1	1	II

ISSN 2278-7763

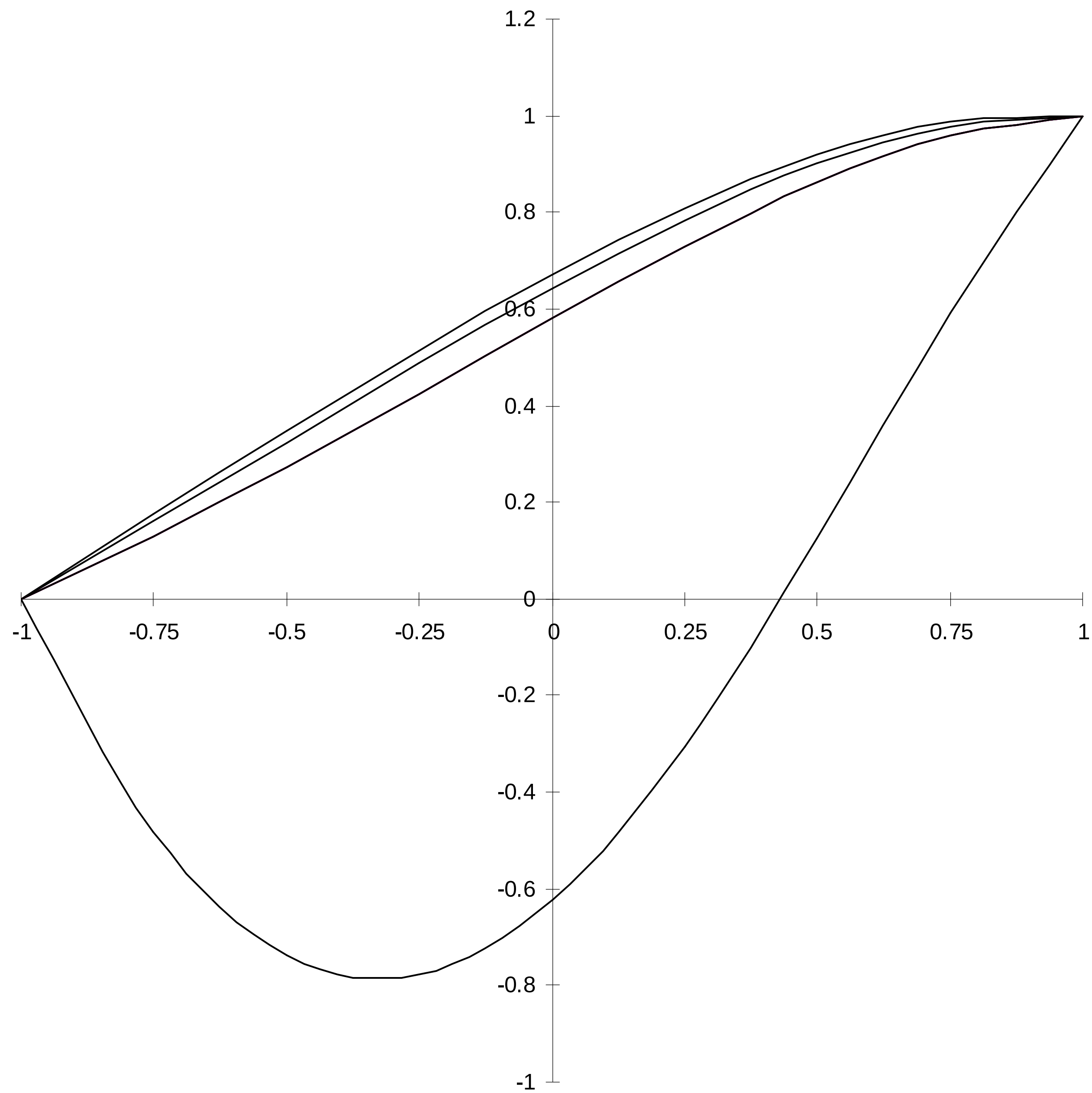

Fig.1. Velocity distribution versus y when Pr=5, Re=1,Ec=0.1,ε=0,λ=1,ω=2, ωt=π/4.

Unsteady Flow

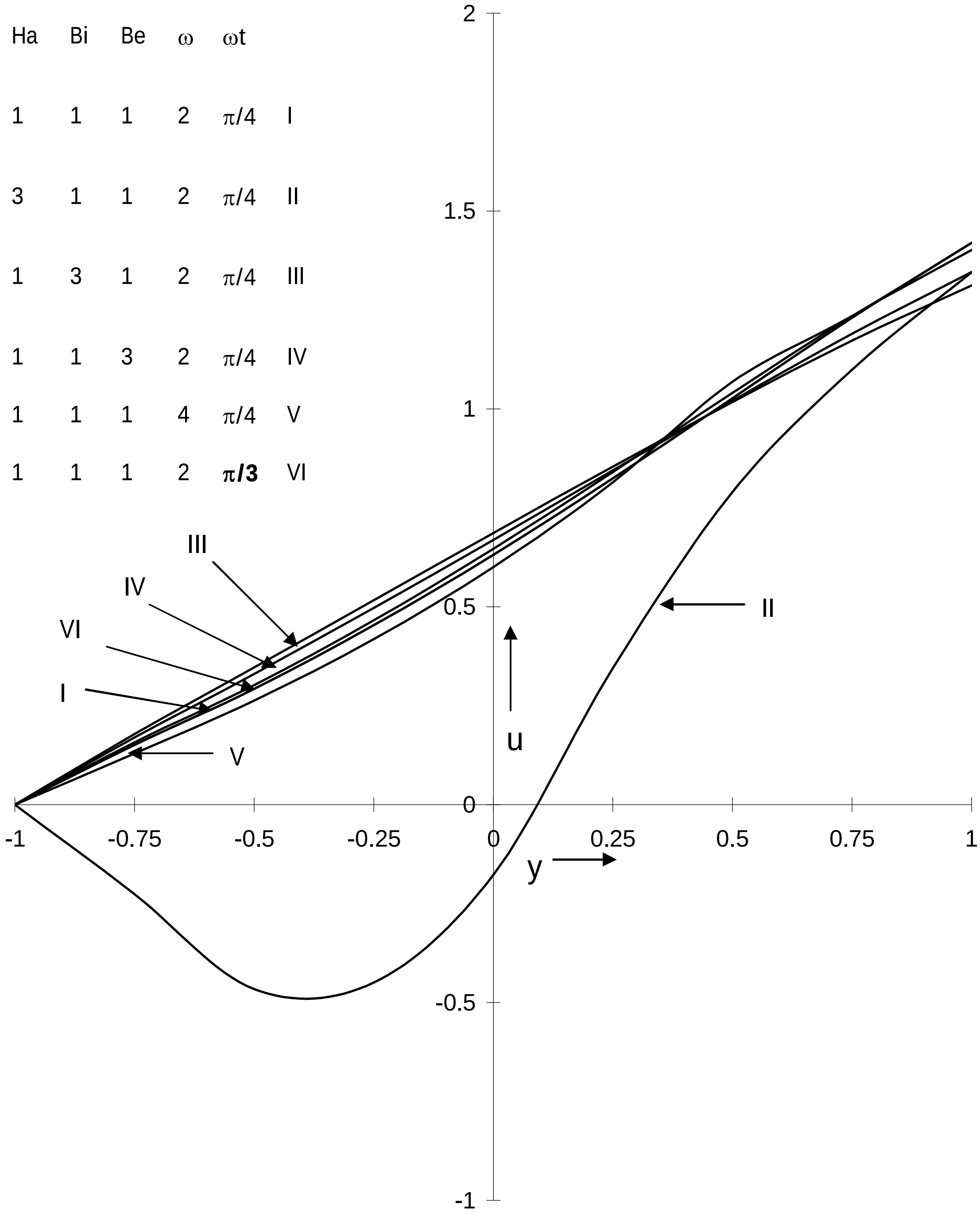

Fig.2. Velocity distribution versus y when Pr=5,Re=1, S=1,Ec=0.1,ε=0.1,λ=1.

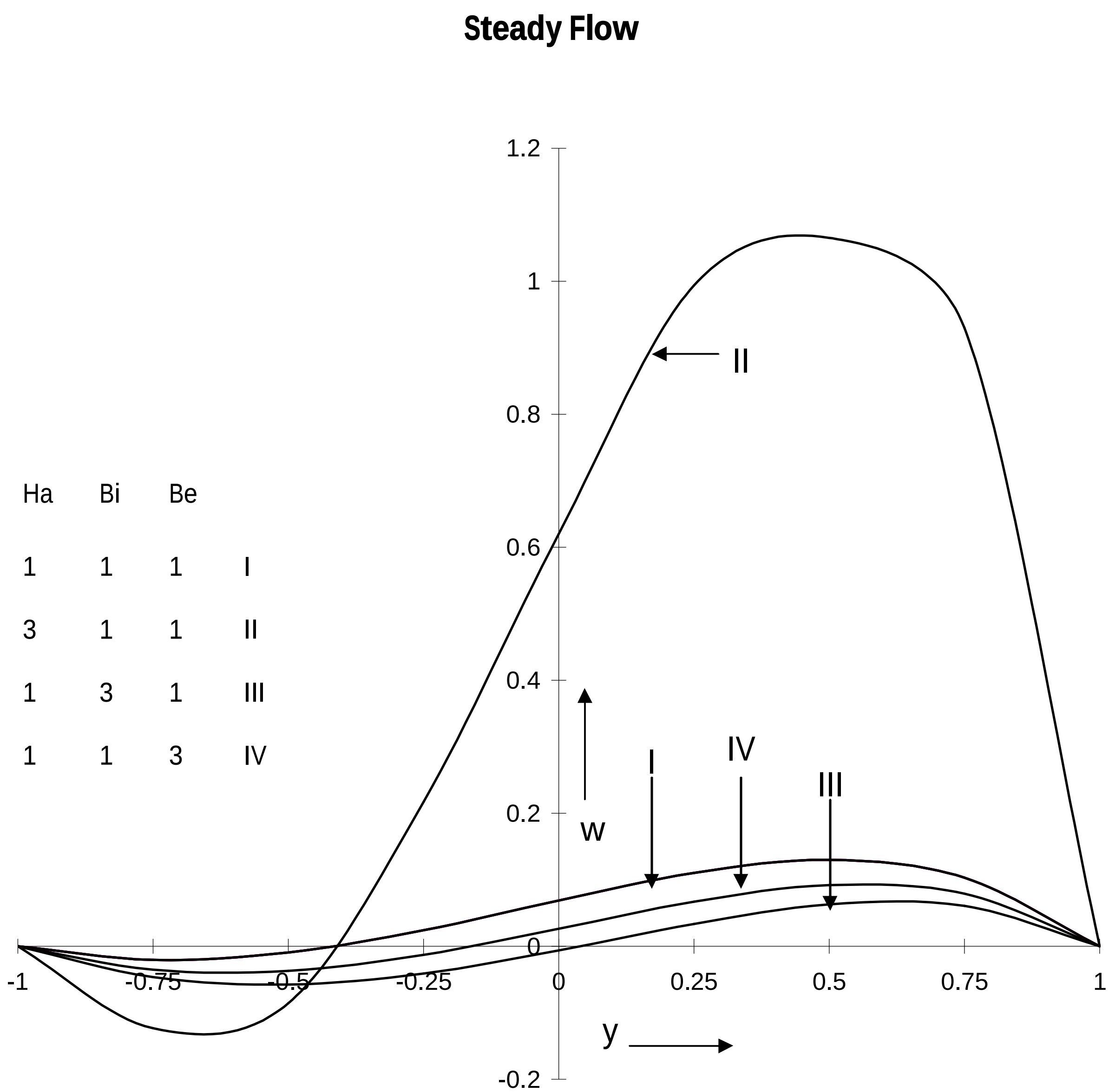

Fig.3. Velocity distribution versus y when Pr=5, Re=1,Ec=0.1,ε=0,λ=1, ω=2, ωt=π/4.

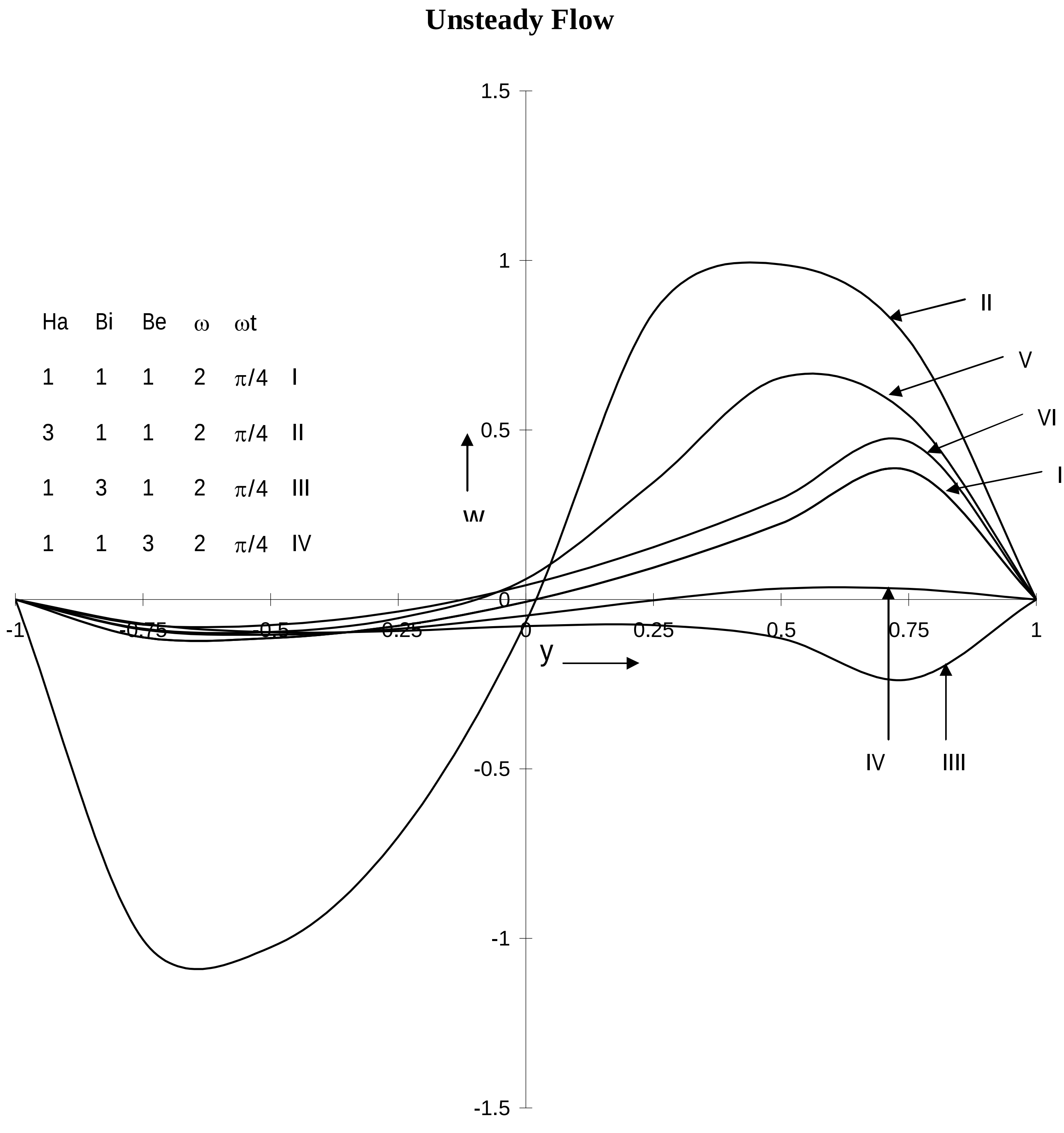

Fig.4.Velocity distribution versus y when Pr=5,Re=1, ε=0.1, λ=1, S=1, Ec=0.1 .

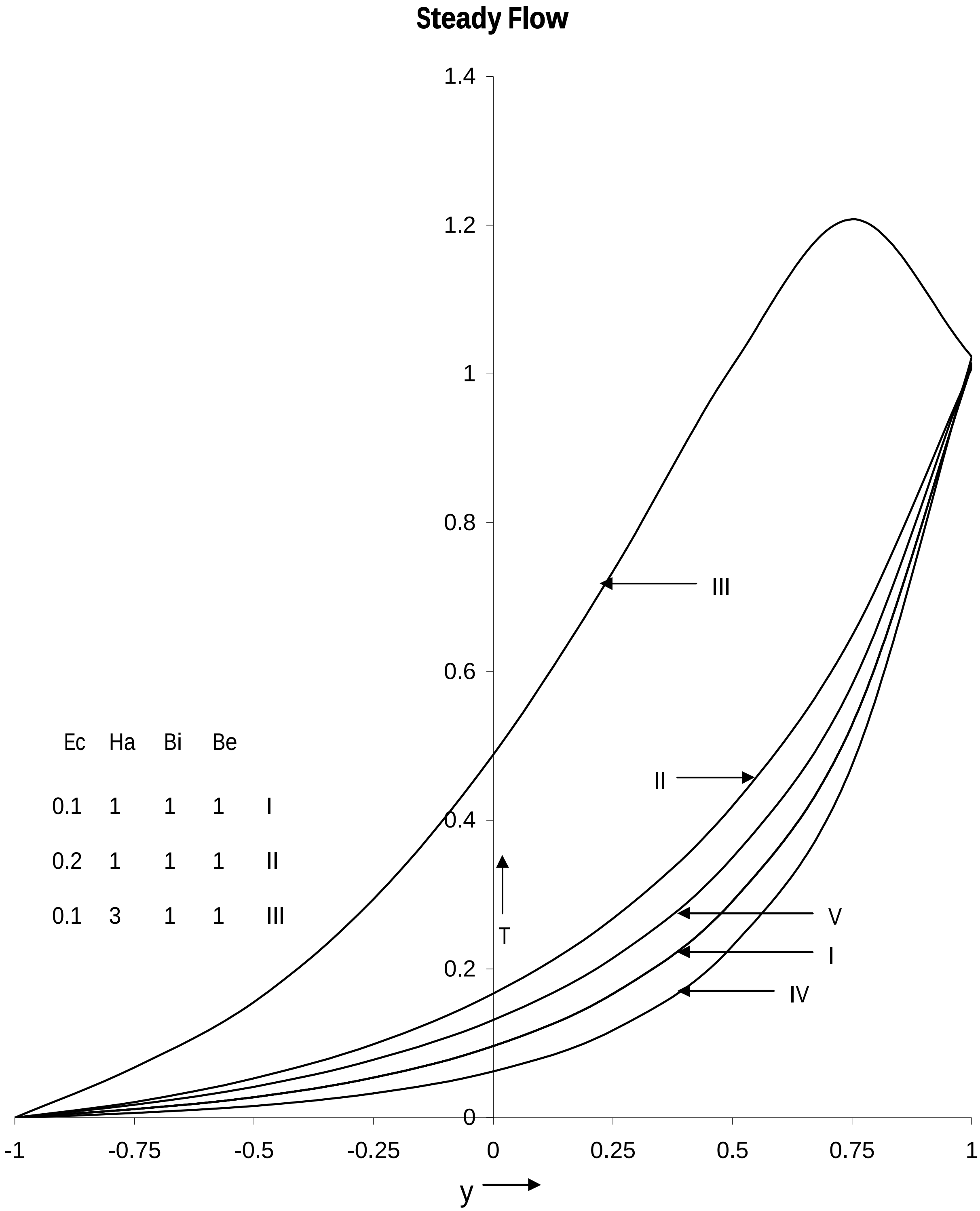

Fig.5. Temperature distribution versus y when Pr=5,Re=1,ε=0,λ=1,S=1,ω=2, ωt=π/4.

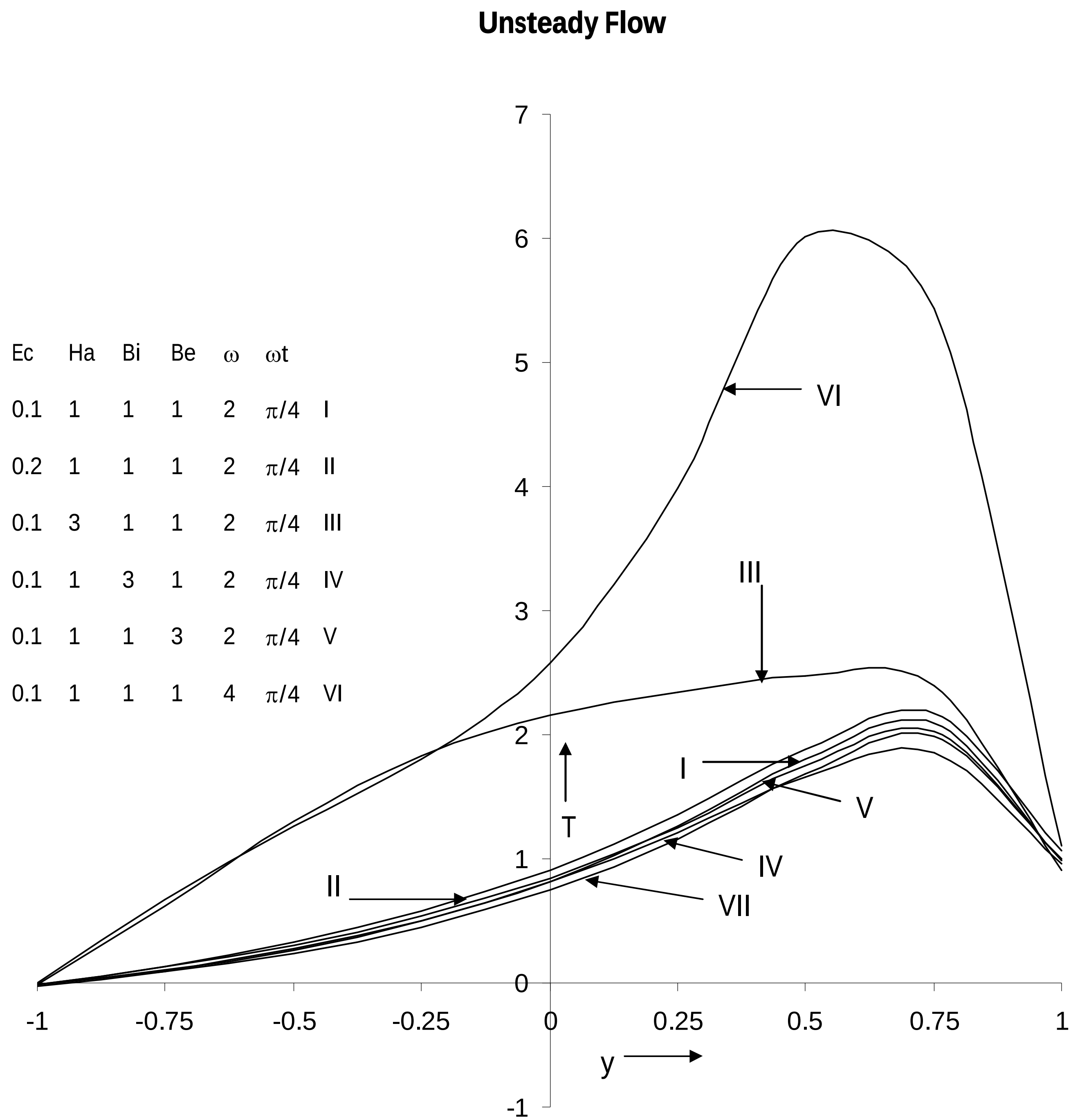

Fig.6. Temperature distribution versus y when Pr=5,Re=1,ε=0.1,λ=1,S=1.

ISSN 2278-7763

Table-1(a). Numerical values of Skin-friction coefficient at lower plate for various values of physical parameters .

Re	Ha	Bi	Be	ω	A	λ	C_f	
							ε=0	ε=0.1
1	1	1	1	2	1	1	-0.8979	-0.9768
2	1	1	1	2	1	1	-1.7564	-1.9963
1	3	1	1	2	1	1	-0.8979	-0.9862
1	1	3	1	2	1	1	-0.74933	-0.8283
1	1	1	3	2	1	1	-0.6814	-0.7603
1	1	1	1	4	1	1	-1.2489	-1.3763
1	1	1	1	2	2	1	-1.8114	-1.9943
1	1	1	1	2	1	2	-3.1039	-3.1626

Table-1(b).Numerical values of Skin-friction coefficient at upper plate for various values of physical parameters

Re	Ha	Bi	Be	ω	A	λ	C_f	
							ε=0	ε=0.1
1	1	1	1	2	1	1	3.9037	4.4923
2	1	1	1	2	1	1	128.99	149.1422
1	3	1	1	2	1	1	3.9037	4.4830
1	1	3	1	2	1	1	3.8360	4.4246
1	1	1	3	2	1	1	3.8051	4.3936
1	1	1	1	4	1	1	17.2236	19.7449
1	1	1	1	2	2	1	7.1956	8.4053
1	1	1	1	2	1	2	18.0925	18.4829

Table-2(a). Numerical values of Nusselt number at the lower plate for various values of physical parameters

Re	Ha	Bi	Be	ω	A	λ	Pr	Nu	
								ε=0	ε=0.1
1	1	1	1	2	1	1	5	0.3361	0.2600
2	1	1	1	2	1	1	5	0.2850	0.2716
1	3	1	1	2	1	1	5	2.0241	1.1431
1	1	3	1	2	1	1	5	1.1691	0.1522
1	1	1	3	2	1	1	5	0.5531	0.5361
1	1	1	1	4	1	1	5	0.3361	-1.3127
1	1	1	1	2	2	1	5	1.0518	0.9531
1	1	1	1	2	1	2	5	-0.0838	-0.0338
1	1	1	1	2	1	1	10	0.4964	0.4118

Table-2(b). Numerical values of Nusselt number at the upper plate for various values of physical parameters

Re	Ha	Bi	Be	ω	A	λ	Pr	Nu	
								ε=0	ε=0.1
1	1	1	1	2	1	1	5	-5.7511	14.4582
2	1	1	1	2	1	1	5	-3707.6493	-6267.8961
1	3	1	1	2	1	1	5	-59.1991	-34.3277
1	1	3	1	2	1	1	5	-1.1531	18.6235
1	1	1	3	2	1	1	5	-9.7040	9.6742
1	1	1	1	4	1	1	5	-5.7511	188.0042
1	1	1	1	2	2	1	5	-26.628	1.4261
1	1	1	1	2	1	2	5	-59993.0905	49443.3821
1	1	1	1	2	1	1	10	1531.682	-11.8521

ISSN 2278-7763

6.Results and Discussion

The computational analysis is carried out to discuss the behavior of the velocity and temperature distributions for different variations in Physical parameters Ec, Ha, Bi, Be, ω and ωt. Figure 1 shows the profiles of velocity component u for various values of y and independent of t i.e. steady flow. It is observed that the velocity component u decreases with the increase of Hartmann number, ion-slip parameter or the Hall parameter.

Figure 2 depicts the profiles of velocity component u for various values of y and depend also on time t (i.e. unsteady part of u) . It is observed that unsteady velocity profile u is increases with increase of ion-slip parameter or the Hall parameter and ωt, but it decreases with the increase of the Hartmann number or frequency .

It is observed from figure 3 that the velocity component w first decreases due to increase of Hartmann number and then increases. It decreases due to increase in ion-slip parameter or The Hall parameter.

Figure 4 shows the unsteady velocity profile w decreases due to increase in ion-slip parameter or The Hall parameter, while it increases due to increase in Hartmann number, frequency or phase angle.

It is noted from figure 5 that fluid temperature decreases with the increase in ion-slip parameter while it increases with the increase in the Hartmann number, the Eckert number or the Hall parameter .

It is observed from figure 6 that fluid temperature increases with the increase of the Hartmann number, the Eckert number or frequency ,while it decreases with the increase of ion-slip parameter, Hall parameter or phase angle.

It is seen from Table-1(a) that skin-friction coefficient at the lower plate increases with the increase in Bi or Be while it decreases with the increase in Re, Ha, ω , A or λ .The skin-friction coefficient at the upper plate increases with the increase in Re, ω , A or λ while it decreases with the increase in Ha,Bi or Be in unsteady case.

It is seen from Table-2(a) that Nusselt number at the lower plate increases with the increase in Re, Ha, Be, Pr or A, while it decreases with the increases in Bi, ω or λ .The Nusselt number at the upper plate increases with the increase in Bi, ω or λ while it decreases with the increase in Re, Ha, Be, A or Pr in unsteady case.

References

1.Abo-El-Dahab,E. “The effect of Hall current on the steady Hartmann flow subjected to a uniform suction and injection at the bounding plates.”.M.S.thesis,Department of Mathematics, Helwan University,Egypt,1993.

2.Alpher,R. “ Heat transfer in magnetohydrodynamic flow between parallel plates.”Int. J. Heat and Mass Transfer, Vol.3, 1961, pp.108-112..

3.Attia,H. “The unsteady state with heat transfer, taking the Hall effect into consideration , while neglecting ion-slip.” Canadian J.Phys., Vol.76, 1998, pp.739-760 .

4.Attia,H. “Unsteady Couette flow with heat transfer considering Ion-Slip”,Turk J. Phys.,Vol.29,2005,pp.379-388.

5.Cramer,K. and Pai,S. “Megnetofluiddynamics for Engineers and Applied Physicsts,” McGraw-Hill,New York, 1973.

6.Hartmann, J. and Lazarus, F. “The influence of a transverse uniform magnetic field on the flow of a conducting fluid between two infinite parallel, stationary and insulated plates.” Mat. Fys. Medd., Vol.15, 1937, pp.6- 12 .

7.Nigam,S.and Singh,S. “Heat transver by laminar flow between parallel plates under the action of transvers magnetic field . ” Quart. J. Mech. Appl. Math., Vol.13, 1960,pp.85-97 .

8.Schilichting,H. “ Boundary Layer Theory”,McGraw-Hill,New York, 2004.

9.Sharma,P.R. and Mishra,Upendra “Pulsatile MHD flow and heat transfer through a porous channel” .Bull.Pure Appl. Sciences, India, Vol.21E,2002,pp.93-100.

10.Sharma,P.R. and Chaturvedi, Reena. “Unsteady flow and heat transfer of an electrically conducting viscous incompressible fluid between two non -conducting parallel plates under uniform transvers magnetic field ”. ‘Ganita Sandesh’ J.Raj.Ganita Parishad , India,Vol.17,2003,pp.9-14.

11.Sharma,P.R.;Chawla,Seema and Singh, Ranjeet “. Unsteady plane Poiseuille flow and heat transfer in the presence of oscillating temperature of the lower plate”. Indian J. Theo.Phys., India, Vol.52, 2004, pp.215-226.

12.Sharma,P.R. and Yadav,G.R. “Pulsatile MHD flow and heat transfer through a porous channel with Ohmic effect” . Bull.Pure Appl. Sciences, India, Vol.25E,2006,pp.499-509.

13.Sharma,P.R. and Metha,Ruchika “MHD unsteady slip flow and heat transfer in a channel with slip at the permeable boundaries”. J. International Academy of Physical sciences ,India ,Vol.13, 2009, pp.73-92.

14.Soundalgerkar,V.;Vighnesam,N. and Takhar,H. “The effect of the Hall currents on the steady MHD Couette flow with heat transfer.” IEEE Trans. Plasma Sci.,Vol.7,1979,pp.178-187.

15.Soundalgekar,V.and Uplekar,A. “The effect of the Hall currents on the steady MHD Couette flow with heat transfer and variable plate temperatures .”IEEE Trans.Plasma Sci.,Vol.14,1986, pp.579-599.

16.Sutton,G. and Sherman,A. “Engineering Magnetohydrodynamics”, McGraw-Hill,New York,1965.

17.Tani,I. “The Hall effect on the steady motion of an electrically conducting and viscous fluids in channels.” J.of Aerospace Sci.,Vol.29,1962,pp.287-298.

18.Tao,I. “Heat transfer in magnetohydrodynamic flow between parallel plates ”J.of Aerospace Sci.,Vol.27,1960,pp.334-340.

Appendix

$$A_1 = A_3 - \frac{A}{\lambda^2 R_3},\qquad A_2 = \frac{(2A-\lambda)}{2\lambda\ Sin(\lambda \operatorname{Re})},$$

$$A_3 = \frac{A}{\lambda} - A_2 e^{-\lambda \operatorname{Re}},\qquad A_5 = A_7 + \frac{(B_1 - A_3)}{2\lambda\ \operatorname{Re}} + \frac{B_1 A}{\lambda^3 \operatorname{Re}},$$

$$A_7 = -A_8 e^{-\lambda \operatorname{Re}} + A_9 e^{-\lambda \operatorname{Re}} - A_{10} + A_{11},\qquad A_8 = \frac{A_9 Sin(\lambda \operatorname{Re}) - A_{11}}{Sin(\lambda \operatorname{Re})},$$

$$A_9 = \frac{B_1 A_2}{\lambda},\qquad A_{10} = \frac{B_1 A}{2\lambda^2},$$

$$A_{11} = \frac{B_1 A}{\lambda^3 \operatorname{Re}} - \frac{B_1 A_3}{\lambda},\qquad A_{12} = A_{14} - \frac{B_2 A_3}{\lambda^2 \operatorname{Re}} - \frac{A}{\lambda^3 \operatorname{Re}^2},$$

$$A_{13} = A_{15} - \frac{B_2 A_2}{(1+\lambda \operatorname{Re})},\qquad A_{14} = -A_{15} e^{\lambda \operatorname{Re}} - A_{16} - A_{17},$$

$$A_{15} = -\frac{-A_{17}}{Sin(\lambda \operatorname{Re})},\qquad A_{16} = \frac{A}{2\lambda^2},$$

$$A_{17} = \frac{B_2 A_3}{\lambda} + \frac{A}{\lambda^3 \operatorname{Re}},\qquad A_{18} = -e^{R_1}\left[A_{19} e^{-R_2} + \frac{iA_2 \lambda^2 \operatorname{Re}}{\omega} e^{-\lambda \operatorname{Re}} + \frac{iA}{\omega}\right],$$

$$A_{19} = \frac{1 - \frac{iA_2 \lambda^2 \operatorname{Re}}{\omega}(e^{\lambda \operatorname{Re}} - e^{2R_1 - \lambda Re}) - \frac{iA}{\omega}(1 - e^{2R_1})}{(e^{R_2} - e^{2R_1 - R_2})},$$

$$A_{22} = -e^{R_1}(A_{23} e^{-R_2} + (iA_{26} - A_{25} - A_{27}) e^{-\lambda \operatorname{Re}} + (A_{28} + iA_{29}) + (A_{32} + iA_{33}) e^{-RR_1} cos(IR_1) + (A_{34} + iA_{35}) e^{RR_1}$$
$$sin(IR_1) + (A_{38} + iA_{39}) e^{-RR_2} cos(IR_2) + (A_{40} + iA_{41}) e^{-RR_2} sin(IR_2),$$

$$A_{23} = -e^{-R_2}[(A_{22} e^{R_1} + (iA_{24} - A_{25}) e^{\lambda \operatorname{Re}} + (iA_{26} - A_{27}) e^{\lambda \operatorname{Re}} + (A_{28} i - A_{29}) + (A_{32} + iA_{33}) e^{RR_1} cos(IR_1) +$$
$$(A_{34} + iA_{35}) sin(IR_1) e^{RR_1} + (A_{38} + iA_{39}) e^{RR_2} cos(IR_2) + (A_{40} + iA_{41}) e^{RR_2} sin(IR_2)],$$

$$A_{24}=\frac{\lambda^2 A_8+\lambda A_9}{\omega},\ A_{25}=\frac{E_1\lambda}{\omega},\ A_{26}=\frac{A_9\lambda^2\,\mathrm{Re}}{\omega},\ A_{27}=\frac{A_9\lambda^3\,\mathrm{Re}}{\omega^2},$$

$$A_{28}=\frac{2\lambda A_{10}}{\omega},\ A_{29}=\frac{2\lambda^2 A_{10}}{\omega^2},\ A_{30}=(RR_1^2-IR_1^2-\lambda\,\mathrm{Re}\,RR_1),$$

$$A_{31}=(IR_1RR_1-\lambda\,\mathrm{Re}\,IR_1-\omega\,\mathrm{Re}),\quad A_{32}=\frac{B_1\,\mathrm{Re}(RA_{18}A_{30}-IA_{18}A_{31})}{A_{30}^2+A_{31}^2},$$

$$A_{33}=\frac{-B_1\,\mathrm{Re}(RA_{18}A_{31}+IA_{18}A_{30})}{A_{30}^2+A_{31}^2},\quad A_{34}=\frac{B_1\,\mathrm{Re}(RA_{18}A_{31}+IA_{18}A_{30})}{A_{30}^2+A_{31}^2},$$

$$A_{35}=\frac{-B_1\,\mathrm{Re}(IA_{18}A_{37}-RA_{18}A_{39})}{A_{30}^2+A_{31}^2},\quad A_{36}=(RR_2^2-IR_2^2-\lambda\,\mathrm{Re}\,RR_1),$$

$$A_{37}=(IP_2RP_2-\lambda\,\mathrm{Re}\,IP_2-\omega\,\mathrm{Re}),\quad A_{38}=\frac{B_1\,\mathrm{Re}(RA_{19}A_{36}-IA_{19}A_{37})}{A_{36}^2+A_{37}^2},$$

$$A_{39}=\frac{-B_1\,\mathrm{Re}(RA_{19}A_{37}+IA_{19}A_{36})}{A_{36}^2+A_{37}^2},\quad A_{40}=\frac{B_1\,\mathrm{Re}(RA_{19}A_{37}+IA_{19}A_{36})}{A_{36}^2+A_{37}^2},$$

$$A_{41}=\frac{-B_1\,\mathrm{Re}(IA_{19}A_{37}+RA_{19}A_{36})}{A_{36}^2+A_{37}^2},$$

$$A_{42}=A_{48}-[(A_{43}-A_{49})e^{R_1-R_2}+iA_{44}e^{(R_1-\lambda\,\mathrm{Re})}+\{iA_{45}+i(\frac{\lambda}{\omega}A_{45}+A_{47})-A_{46}\}e^{R_1}],$$

$$A_{43}=A_{49}+\frac{iA_{44}(e^{\lambda\,\mathrm{Re}}-e^{(2R_1-\lambda\,\mathrm{Re})})+iA_{45}(1+e^{2R_1})+\{i(\frac{\lambda}{\omega}A_{45}+A_{47})-A_{46}\}(1-e^{2R_1})\}}{(e^{(2R_1-R_2)}-e^{R_2})},$$

$$A_{44}=\frac{(A_{15}\lambda^2\,\mathrm{Re}-B_2E_1)}{\omega},\ A_{45}=\frac{2\lambda A_{16}}{\omega},\ A_{46}=\frac{-2\lambda^2A_{16}}{\omega^2}=\frac{\lambda}{\omega}A_{45},$$

$$A_{47}=\frac{iA_{17}\lambda+B_2E_2}{\omega},\ A_{48}=\frac{B_2A_{18}}{\frac{R_1^2}{\mathrm{Re}}-\lambda R_1-i\omega},\ A_{49}=\frac{B_2A_{19}}{\frac{R_2^2}{\mathrm{Re}}-\lambda R_2-i\omega},$$

$$A_{50}=-e^{-R_3}(A_{51}e^{R_4}-A_{52}e^{\lambda\,\mathrm{Re}}-A_{53})$$

$$A_{51}=\frac{e^{2R_3}+A_{52}(e^{2R_3-\lambda\,\mathrm{Re}}-e^{\lambda\,\mathrm{Re}})+A_{53}(e^{2R_3}-1)}{(e^{2R_3-R_4}-e^{R_4})},\quad A_{52}=\frac{Ec\lambda^2A_2^2\,\mathrm{Pr}\,\mathrm{Re}}{2\lambda^2\,\mathrm{Re}(2-\mathrm{Pr})+S\,\mathrm{Pr}},$$

$$A_{53} = \frac{EcA}{\operatorname{Re} S\lambda},$$

$$A_{54} = -A_{56}e^{R_3-R_4} + (A_{64}+A_{66})e^{R_3-2\lambda\operatorname{Re}} - A_{65}e^{R_3-2\lambda\operatorname{Re}} - A_{67}e^{R_3-\lambda\operatorname{Re}}$$
$$+(A_{68}+A_{69})e^{R_3-\lambda\operatorname{Re}} + (-A_{70}+A_{71}+A_{72}-A_{73}+A_{74}+A_{75})e^{R_3}$$

$$A_{56} = \frac{(A_{64}+A_{66})(e^{2R_3-2\lambda\operatorname{Re}} - e^{2\lambda\operatorname{Re}}) - A_{65}(e^{2R_3-2\lambda\operatorname{Re}} + e^{2\lambda\operatorname{Re}}) - A_{67}(e^{2R_3-\lambda\operatorname{Re}} + e^{\lambda\operatorname{Re}}) + (A_{68}+A_{69})(e^{2R_3-\lambda\operatorname{Re}} - e^{\lambda\operatorname{Re}})}{(e^{2R_3-R_4} - e^{R_4})}$$
$$\underline{-(A_{70}+A_{73})(e^{2R_3}+1) + (A_{71}+A_{72}+A_{74}+A_{75})(e^{2R_3}-1)},$$

$$A_{57} = Ec\left[2A_2A_8\lambda^2\operatorname{Re} + 2A_2A_9 + \frac{A_2^2B_2}{Be}\right], \qquad A_{58} = Ec\left(2A_2A_9\lambda^2\operatorname{Re}\right),$$

$$A_{59} = 2Ec\left[2A_2A_{10} + AA_9 + \frac{A_2AB_2}{\lambda^2 Be}\right], \qquad A_{60} = 2Ec\left[A_2A_{11}\lambda + AA_8 + \frac{AA_9}{\lambda\operatorname{Re}} + \frac{A_2A_3B_2}{\operatorname{Re}}\right],$$

$$A_{61} = \frac{4EcAA_{10}}{\lambda\operatorname{Re}}, \qquad A_{62} = \frac{4EcA^2B_2}{\lambda^2},$$

$$A_{63} = Ec\left[\frac{2AA_{11}}{\operatorname{Re}\lambda} + \frac{A_3^2B_2}{Be}\right], \qquad A_{64} = \frac{A_{57}}{2\lambda^2\operatorname{Re}\left(\frac{2}{\operatorname{Pr}}-1\right)+S},$$

$$A_{65} = \frac{A_{58}}{(4\lambda^2\operatorname{Re} - 2\lambda^2R_2^2\operatorname{Pr} + S\operatorname{Re}\operatorname{Pr})}, \qquad A_{66} = \frac{A_{58}\lambda\operatorname{Re}(4-\operatorname{Pr})}{(4\lambda^2\operatorname{Re} - 2\lambda^2\operatorname{Re}^2\operatorname{Pr} + S\operatorname{Re}\operatorname{Pr})^2},$$

$$A_{67} = \frac{A_{59}(2\lambda/\operatorname{Pr} - \lambda)}{(2\lambda^2\operatorname{Re} - 2\lambda^2\operatorname{Re}^2\operatorname{Pr} + S\operatorname{Re}\operatorname{Pr})^2}, \qquad A_{68} = \frac{A_{59}\left(\frac{2\lambda}{\operatorname{Pr}} - \lambda\right)}{(2\lambda^2\operatorname{Re} - \lambda^2\operatorname{Re}^2\operatorname{Pr} + S\operatorname{Re}\operatorname{Pr})^2},$$

$$A_{69} = \frac{A_{60}}{\lambda^2\operatorname{Re}\left(\frac{1}{\operatorname{Pr}}-1\right)+S}, \qquad A_{70} = \frac{A_{61}}{S},$$

$$A_{71} = \frac{A_{61}}{S^2}, \qquad A_{72} = \frac{A_{62}}{S},$$

$$A_{73} = \frac{2A_{62}}{S^2}, \qquad A_{74} = \frac{2A_{62}}{S^2\lambda\operatorname{Re}\operatorname{Pr}},$$

$$A_{75} = \frac{A_{63}}{S} \quad , \qquad A_{76} = -A_{77}e^{R_6 - R_5} + p_1 e -^{R_5} \quad ,$$

$$A_{77} = \frac{p_1 - q_1 e^2 R_5}{(e^{R_6} - e^{2R_5 - R_6})} \quad , \qquad A_{78} = 2EcA_2 \lambda A_{18} R_1 \quad ,$$

$$A_{79} = 2EcA_2 \lambda A_{19} \quad , \qquad A_{80} = 2EcE_1 \lambda^2 \operatorname{Re} A_2 ,$$

$$A_{81} = \frac{2EcAA_{18}R_1}{\lambda \operatorname{Re}} \quad , \qquad A_{82} = 2EcAE_1 \quad ,$$

$$A_{83} = E_1 EcA \quad , \qquad A_{84} = \lambda A_{50} R_3 ,$$

$$A_{85} = \lambda A_{51} R_4 \quad , \qquad A_{86} = A_{52} \lambda^2 R_3 \quad ,$$

$$A_{87} = \frac{A_{78}}{\frac{(R_1 + \lambda \operatorname{Re})^2}{\operatorname{Re}\Pr} - \lambda (R_1 + \lambda \operatorname{Re}) + (S - i\omega)} \quad , \qquad A_{88} = \frac{A_{79}}{\frac{(R_2 + \lambda \operatorname{Re})^2}{\operatorname{Re}\Pr} - \lambda (R_2 + \lambda \operatorname{Re}) + (S - i\omega)} \quad ,$$

$$A_{89} = \frac{A_{82}}{\frac{R_2^2}{\operatorname{Re}\Pr} - \lambda R_2 + (S - i\omega)} \quad , \qquad A_{90} = \frac{A_{50}\lambda R_3}{\frac{R_3^2}{\operatorname{Re}\Pr} - \lambda \operatorname{Re} + (S - i\omega)} \quad ,$$

$$A_{91} = \frac{A_{51}\lambda R_4}{\frac{R_4^2}{\operatorname{Re}\Pr} - \lambda R_4 + (S - i\omega)} \quad , \qquad A_{92} = \frac{A_{52}\lambda^2 \operatorname{Re} + iA_{83}}{\lambda^2 \operatorname{Re}\left(\frac{1}{\Pr} - 1\right) + (S - i\omega)} \quad ,$$

$$A_{93} = -\left(A_{92} e^{R_5 - R_6} + p_2\right) \quad , \qquad A_{94} = \frac{(p_2 e^{2R_5} - q_2)}{(e^{2R_5 - R_6} - e^{R_6})} \quad ,$$

$$RA_{95} = \frac{\{(RD_{61} + RD_{69})C_3 + (ID_{61} + ID_{69})C_4\}\left(\frac{2RR_2}{\operatorname{Re}\Pr} - \lambda\right) IR_2}{C_3^2 + C_4^2} \quad ,$$

$$IA_{95} = -\frac{\{(RD_{61} + RD_{69})C_4 + (ID_{61} + ID_{69})C_3\}\left(\frac{2RR_2}{\operatorname{Re}\Pr} - \lambda\right) IR_2}{C_3^2 + C_4^2} \quad ,$$

$$RA_{96} = \frac{-\{(RD_{61} + RD_{69})C_4 + (ID_{61} + ID_{69})C_3\}(S + \frac{RR_2^2}{\operatorname{Re}\Pr} - \lambda RR_2 - \frac{IR_2^2}{\operatorname{Re}\Pr})}{C_3^2 + C_4^2}$$

$$\underline{+(RD_{61}+RD_{69})C_3+(ID_{61}+ID_{69})C_4\}\omega+\{RD_{61}+RD_{69})C_4+(ID_{61}+ID_{69})C_3\}\omega}$$
,

$$IA_{96}=\frac{-(RD_{61}+RD_{69})C_4+(ID_{61}+ID_{69})C_3\}\omega+\{RD_{61}+RD_{69})C_4+(ID_{61}+ID_{69})C_3\}\{S+\frac{RR_2^2}{\mathrm{Re}\,\mathrm{Pr}}-\lambda RR_2-\frac{IR_2^2}{\mathrm{Re}\,\mathrm{Pr}}\}}{C_3^2+C_4^2},$$

$$A_{97}=\frac{D_{62}}{\left\{\frac{4\lambda^2\,\mathrm{Re}}{\mathrm{Pr}}-2\lambda^2\,\mathrm{Re}+S\right\}-i\omega},\qquad A_{98}=\frac{D_{63}}{\left\{\frac{4\lambda^2\,\mathrm{Re}}{\mathrm{Pr}}-2\lambda^2\,\mathrm{Re}+S\right\}-i\omega}$$

$$A_{98}=\frac{D_{63}}{\left\{\frac{4\lambda^2\,\mathrm{Re}}{\mathrm{Pr}}-2\lambda^2\,\mathrm{Re}+S\right\}-i\omega},\qquad A_{99}=\frac{D_{63}\left(\frac{4\lambda}{\mathrm{Pr}}-\lambda\right)}{\left\{\frac{4\lambda^2\,\mathrm{Re}}{\mathrm{Pr}}-2\lambda^2\,\mathrm{Re}+S-i\omega\right\}^2}$$

$$A_{100}=\frac{D_{64}}{\left\{\frac{\lambda^2\,\mathrm{Re}}{\mathrm{Pr}}-\lambda^2\,\mathrm{Re}+S\right\}-i\omega},\qquad A_{101}=\frac{D_{65}}{\left\{\frac{R_1^2}{\mathrm{Re}\,\mathrm{Pr}}-\lambda R_1+S\right\}-i\omega}$$

$$A_{102}=\frac{D_{66}}{\left\{\frac{R_2^2}{\mathrm{Re}\,\mathrm{Pr}}-\lambda R_2+S\right\}-i\omega},\qquad A_{103}=\frac{D_{71}}{(S-i\omega)}$$

$$A_{103}=\frac{D_{72}}{(S-i\omega)},\qquad A_{105}=\frac{D_{72}}{(S-i\omega)}$$

$$A_{105}=\frac{\lambda D_{72}}{(S-i\omega)},$$

$$B_1=\frac{(1+BiBe)}{\mathrm{Re}\{(1+BiBe)^2+Be^2\}},\qquad B_2=\frac{Be}{\mathrm{Re}\{(1+BiBe)^2+Be^2\}}$$

$$C_1=\left[-(A_{25}+A_{27})(e^{2RR_1-\lambda\mathrm{Re}}\cos(IR_1)-e^{\lambda\mathrm{Re}})-A_{24}e^{2RR_1-\lambda\mathrm{Re}}\sin(IR_1)\right],$$

$$C_2=-A_{26}e^{2R_1-\lambda\mathrm{Re}}Sin(IR_1),$$

$$C_3=\{A_{32}e^{RR_1}(cos(2IR_1)-1)-A_{33}e^{RR_1}Sin(2IR_1)\}cos(IR_1),$$

$$C_4=(A_{34}(cos(2IR_1)+1)-A_{35}sin(2IR_1))e^{RR_1}sin(IR_1),$$

$$C_5 = A_{38}(e^{2RR_1-RR_2}cos(2IR_1) - e^{RR_2}) - A_{39}e^{2RR_1-RR_2}sin(2IR_1) \quad ,$$

$$C_6 = \{A_{40}(e^{2RR_1-RR_2}cos(2IR_1) - e^{RR_2}) - A_{41}e^{2RR_1-RR_2}sin(2IR_1)\}sin(IR_2) \quad ,$$

$$C_7 = e^{2RR_1-RR_2}cos(2IR_1 - IR_2) - e^{RR_2}cos(IR_2) \quad ,$$

$$C_8 = (e^{2RR_1-RR_2}\sin(2IR_1 - IR_2) - e^{RR_2}sin(IR_2)) \quad ,$$

$$C_9 = \left(A_{24}(e^{2RR_1-\lambda \mathrm{Re}}cos(2IR_1) - e^{\lambda \mathrm{Re}}) + (A_{25} + A_{27})(e^{2RR_1-\lambda \mathrm{Re}}sin(IR_2))\right) \quad ,$$

$$C_{10} = A_{26}(e^{\lambda \mathrm{Re}} + e^{RR_1-\lambda \mathrm{Re}}cos(2IR_1)) \quad ,$$

$$C_{11} = (A_{32}sin(2IR_1) + A_{33}(cos(2IR_1) - 1))e^{RR_1}cos(IR_1) \quad ,$$

$$C_{12} = (A_{35}(cos(2IR_1) + 1) + A_{34}\sin(2IR_1))e^{RR_1}\sin(IR_1) \quad ,$$

$$C_{13} = ((A_{38}e^{2RR_1-RR_2}\sin(2IR_1) + A_{39}(e^{2RR_1-RR_2}\cos(2IR_1) - e^{RR_2}))cos(IR_2) \quad ,$$

$$C_{14} = (A_{41}(e^{2RR_1-RR_2}cos(2IR_2) - e^{RR_2}) + A_{40}e^{2RR_1-RR_2}sin(2IR_1))\sin(IR_2) \quad ,$$

$$C_{15} = (A_{44}(e^{\lambda \mathrm{Re}} - e^{2RR_1-\lambda \mathrm{Re}}cos(2IR_2) - A_{45}e^{2RR_1}sin(2IR_1) + A_{46}(e^{2RR_1}cos(IR_1) - 1)$$
$$+A_{47}(1 - e^{2RR_1}cos(2IR_1))) \quad ,$$

$$C_{16} = -A_{44}(e^{2RR_1-\lambda \mathrm{Re}}sin(2IR_2)) + A_{45}(e^{2RR_1}cos(2IR_1) + 1) - A_{46}e^{2RR_1}sin(2IR_1)$$
$$-A_{47}(1 - e^{2RR_1}cos(2IR_1)) \quad ,$$

$$C_{17} = (-e^{RR_2}cos(IR_2) + e^{2RR_1-RR_2}cos(2IR_1 - IR_2))$$

$$C_{18} = (-e^{RR_2}\sin(IR_2) + e^{2RR_1-RR_2}sin(2IR_1 - IR_2)) \quad ,$$

$$RD_1 = \lambda \mathrm{Re}\, A_2(RA_{22}RR_1 - IA_{22}IR_1) \quad , \qquad ID_1 = \lambda \mathrm{Re}\, A_2(RA_{22}IR_1 + IA_{22}RR_1) \,,$$

$$RD_2 = \lambda \mathrm{Re}\, A_2(RA_{23}RR_2 - IA_{23}IR_2) \quad , \qquad ID_2 = \lambda \mathrm{Re}\, A_2(RA_{23}IR_2 + IA_{23}RR_2) \,,$$

$$RD_3 = A_2\lambda^2 \mathrm{Re}^2(A_{25} + A_{27}) \quad , \qquad ID_3 = A_2\lambda^2 \mathrm{Re}^2(A_{26} - A_{24}) \,,$$

$$RD_4 = -A_2\lambda^2 \mathrm{Re}^2 IA_{26} \quad , \qquad ID_4 = -A_2 RA_{26}\lambda^2 \mathrm{Re}^2 \,,$$

$$RD_5 = -A_2\lambda \mathrm{Re}\, RA_{28} \quad , \qquad ID_5 = A_2\lambda \mathrm{Re}\, IA_{28} \,,$$

$RD_6 = A_2\lambda \operatorname{Re}(RA_{32} - IA_{33})$,

$ID_6 = A_2\lambda \operatorname{Re}(IA_{32} + RA_{33})$,

$RD_7 = A_2\lambda \operatorname{Re}(RA_{34} - IA_{35})$,

$ID_7 = A_2\lambda \operatorname{Re}(IA_{34} + RA_{35})$,

$RD_8 = A_2\lambda \operatorname{Re}(RA_{38} - IA_{39})$,

$ID_8 = A_2\lambda \operatorname{Re}(IA_{38} + RA_{39})$,

$RD_9 = A_2\lambda \operatorname{Re}(RA_{40} - IA_{41})$,

$ID_9 = A_2\lambda \operatorname{Re}(IA_{40} + RA_{41})$,

$RD_{10} = \frac{A}{\lambda}(RA_{22}RR_1 - IA_{22}IR_1)$,

$ID_{11} = \frac{A}{\lambda}(RA_{23}RR_2 - IA_{23}IR_2)$,

$RD_{12} = A\operatorname{Re}(A_{25} + A_{27})$,

$ID_{12} = A\operatorname{Re}(A_{26} - A_{24})$,

$D_{13} = A\operatorname{Re} A_{26}$,

$D_{14} = \frac{A}{\lambda}A_{28}$,

$RD_{15} = \frac{A}{\lambda}A_{32}$,

$ID_{15} = \frac{A}{\lambda}A_{33}$,

$RD_{16} = \frac{A}{\lambda}A_{34}$,

$ID_{16} = \frac{A}{\lambda}A_{35}$,

$RD_{17} = \frac{A}{\lambda}A_{38}$,

$ID_{17} = \frac{A}{\lambda}A_{39}$,

$RD_{18} = \frac{A}{\lambda}A_{40}$,

$ID_{18} = \frac{A}{\lambda}A_{41}$,

$RD_{19} = A_8 \operatorname{Re}(RR_1RA_{18} - IR_1IA_{18})$,

$ID_{19} = A_8\lambda \operatorname{Re}(RR_1IA_{18} + IR_1RA_{18})$,

$RD_{20} = A_8\lambda \operatorname{Re}(RR_2RA_{19} - IR_2IA_{19})$,

$ID_{20} = A_8\lambda \operatorname{Re}(RR_2IA_{19} + IR_2RA_{19})$,

$D_{21} = E_1\lambda^2 \operatorname{Re} A_8$,

$RD_{22} = A_9(RA_{18}RR_1 - IA_{18}IR_1)$,

$ID_{22} = A_9(RA_{18}IR_1 + IA_{18}RR_1)$,

$RD_{23} = \lambda \operatorname{Re} A_9(RR_2RA_{19} - IR_2IA_{19})$,

$ID_{23} = \lambda \operatorname{Re} A_9(RR_2IA_{19} + IR_2RA_{19})$,

$D_{24} = E_1A_9\lambda \operatorname{Re}$,

$RD_{25} = \lambda \operatorname{Re} A_9(RR_1RA_{18} - IR_1IA_{18})$,

$ID_{25} = \lambda \operatorname{Re} A_9(RR_1IA_{18} + IR_1RA_{18})$,

$RD_{26} = \lambda \operatorname{Re} A_9(RR_1RA_{19} - IR_2IA_{19})$,

$ID_{26} = \lambda \operatorname{Re} A_9(RR_2RA_{19} - IR_2IA_{19})$,

$ID_{27} = E_1A_9\lambda^2 \operatorname{Re}^2$

$RD_{28} = 2A_{10}(RR_1 RA_{18} - IR_1 IA_{18})$, $ID_{28} = 2A_{10}(RR_1 IA_{18} + IR_1 RA_{18})$,

$RD_{29} = 2A_{10}(RR_2 RA_{19} - IR_2 IA_{19})$, $ID_{29} = 2A_{10}(RR_2 IA_{19} + IR_2 RA_{19})$,

$D_{30} = 2A_{10}E_1 \lambda \operatorname{Re}$, $RD_{31} = A_{11}(RR_1 RA_{18} - IR_1 IA_{18})$,
$ID_{31} = A_{11}(RR_1 RA_{18} + IR_1 RA_{18})$, $RD_{32} = A_{11}(RR_2 RA_{19} - IR_2 IA_{19})$,
$ID_{32} = A_{11}(RR_2 IA_{19} + IR_2 RA_{19})$, $ID_{33} = E_1 \lambda \operatorname{Re} A_{11}$, $RD_{34} = A_3 RA_{18}$, $ID_{34} = A_3 IA_{18}$,
$RD_{35} = A_3 IA_{19}$, $ID_{35} = A_3 IA_{19}$, $D_{36} = A_3 E_1$, $D_{37} = E_2 A_3$, $RD_{38} = A_2 RA_{18}$,
$ID_{38} = A_2 IA_{18}$, $RD_{39} = A_2 RA_{19}$, $ID_{39} = A_2 IA_{19}$, $D_{40} = A_3 E_1$, $D_{41} = A_3 E_2$,
$RD_{42} = A_2 RA_{18}$, $ID_{42} = A_2 IA_{18}$, $RD_{43} = A_2 RA_{19}$, $ID_{43} = A_2 IA_{19}$, $D_{44} = A_2 E_1$,
$D_{45} = E_2 A_2$, $RD_{46} = \dfrac{A}{\lambda} RA_{18}$, $RD_{47} = \dfrac{A}{\lambda} RA_{19}$, $ID_{47} = \dfrac{A}{\lambda} IA_{19}$, $D_{48} = \dfrac{AE_1}{\lambda}$,
$D_{49} = \dfrac{E_2 A}{\lambda}$, $D_{50} = \lambda \operatorname{Re}_3 A_{55}$, $D_{51} = \lambda R_4 A_{56}$, $D_{52} = 2\lambda^2 \operatorname{Re}(A_{64} + A_{66})$, $D_{53} = A_{65}\lambda$,
$D_{54} = 2A_{65}\lambda^2 \operatorname{Re}$, $D_{55} = A_{67}\lambda$, $D_{56} = 2\lambda_2 \operatorname{Re} A_{67}$, $D_{57} = \lambda^2 \operatorname{Re}(A_{68} + A_{69})$, $D_{58} = 2\lambda A_{72}$,
$D_{59} = \lambda(A_{70} + A_{73})$,

$$RD_{60} = \left(-\frac{2Ec}{\operatorname{Re}} RD_1 - \frac{2Ec}{\operatorname{Re}} RD_{19} - \frac{2Ec}{\operatorname{Re}} RD_{22} + B_3 RD_{38} + B_3 RD_{42} \right),$$

$$ID_{60} = \left(-\frac{2Ec}{\operatorname{Re}} ID_1 - \frac{2Ec}{\operatorname{Re}} ID_{19} - \frac{2Ec}{\operatorname{Re}} ID_{22} + B_3 ID_{38} + B_3 ID_{42} \right),$$

$$RD_{61} = \left(-\frac{2Ec}{\operatorname{Re}} RD_2 - \frac{2Ec}{\operatorname{Re}} RD_{20} - \frac{2Ec}{\operatorname{Re}} RD_{23} + B_3 RD_{39} + B_3 RD_{43} \right),$$

$$ID_{61} = \left(-\frac{2Ec}{\operatorname{Re}} ID_2 - \frac{2Ec}{\operatorname{Re}} ID_{20} - \frac{2Ec}{\operatorname{Re}} ID_{23} + B_3 ID_{39} + B_3 ID_{43} \right),$$

$$RD_{62} = \left(-\frac{2Ec}{\operatorname{Re}} RD_3 - \frac{2Ec}{\operatorname{Re}} RD_{21} - D_{52} - RD_{53} \right),$$

$$ID_{62} = \left(-\frac{2Ec}{\operatorname{Re}} ID_3 - \frac{2Ec}{\operatorname{Re}} ID_{21} + B_3 D_{44} \right),$$

$$RD_{63} = \left(-\frac{2Ec}{\operatorname{Re}} RD_4 - \frac{2EC}{\operatorname{Re}} RD_{27} - \frac{2Ec}{\operatorname{Re}} D_{54} \right),$$

$$ID_{63} = \left(-\frac{2Ec}{\text{Re}} ID_4 - \frac{2EC}{\text{Re}} ID_{27} \right),$$

$$RD_{64} = \left(-\frac{2Ec}{\text{Re}} RD_5 + \frac{2Ec}{\text{Re}} RD_{12} - \frac{2Ec}{\text{Re}} RD_{13} - \frac{2Ec}{\text{Re}} D_{24} + \frac{2Ec}{\text{Re}} ID_{33} - D_{55} - D_{57} \right),$$

$$RD_{64} = \left(-\frac{2Ec}{\text{Re}} ID_5 + \frac{2Ec}{\text{Re}} ID_{12} - \frac{2Ec}{\text{Re}} ID_{13} - \frac{2Ec}{\text{Re}} D_{30} - \frac{2Ec}{\text{Re}} D_{33} + B_3 D_{36} + B_3 D_{40} + B_3 D_{45} + B_3 D_{48} \right),$$

$$RD_{65} = \left(-\frac{2Ec}{\text{Re}} RD_{10} - \frac{2Ec}{\text{Re}} RD_{31} - B_3 RD_{34} \right), \qquad ID_{65} = \left(-\frac{2Ec}{\text{Re}} ID_{10} - \frac{2EC}{\text{Re}} ID_{31} + B_3 ID_{34} \right),$$

$$RD_{66} = \left(-\frac{2Ec}{\text{Re}} RD_{11} - \frac{2EC}{\text{Re}} RD_{32} + B_3 RD_{35} \right), \qquad ID_{66} = \left(-\frac{2Ec}{\text{Re}} ID_{11} - \frac{2Ec}{\text{Re}} ID_{32} + B_3 ID_{35} \right),$$

$$RD_{67} = -\frac{2Ec}{\text{Re}} (RD_{15} RR_1 + RD_{16} IR_1), \qquad ID_{67} = -\frac{2Ec}{\text{Re}} (ID_{15} RR_1 + ID_{16} IR_1),$$

$$RD_{68} = -\frac{2Ec}{\text{Re}} (-RD_{15} IR_1 + RD_{16} RR_1), \qquad ID_{68} = -\frac{2Ec}{\text{Re}} (-ID_{15} IR_1 + RD_{16} RR_1),$$

$$RD_{69} = -\frac{2Ec}{\text{Re}} (RD_{17} RR_2 + RD_{18} IR_1), \qquad ID_{69} = -\frac{2Ec}{\text{Re}} (ID_{17} RR_1 + ID_{18} IR_2),$$

$$RD_{70} = -\frac{2Ec}{\text{Re}} (-RD_{17} IR_2 + RD_{18} RR_2), \qquad ID_{70} = -\frac{2Ec}{\text{Re}} (-ID_{17} IR_2 + ID_{18} RR_2),$$

$$RD_{71} = -\frac{2Ec}{\text{Re}} D_{14} - D_{59}, \quad ID_{71} = B_3 D_{41}, \qquad E_1 = \frac{A_2 \lambda \text{Re}}{\omega}, \qquad E_2 = A/\omega.$$