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Total Dominator Colorings in Graphs

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Abstract

A total dominator coloring of graphs with minimum degree atleast one is a proper coloring of graphs with the extra property that every vertex in the graph properly dominates a color class. The total dominator chromatic number is denoted by χ_{td} (G) and is defined by the minimum number of colors needed in a total dominator coloring of G. In this paper we find χ_{td} (G) for some classes of graphs and obtain a general bound. Also we obtain a characterization for lower and upper bound.

Key words and Phrases: Total domination number, chromatic number and total dominator chromatic number.

2000 Mathematics Subject Classification: 05C15, 05C69

1. Introduction.

All graphs considered in this paper are finite, undirected graphs and we follow standard definitions of graph theory as found in [3].

Let G=(V,E) be a graph of order p with minimum degree at least one. The open neighborhood N(v) of a vertex $v \in V(G)$ consists of the set of all vertices adjacent to v. The closed neighborhood of v is $N[v]=N(v)\cup\{v\}$.For a set $S \subseteq V$,the open neighborhood N(S) is defined to be $\bigcup_{v\in S} N(v)$,and the closed neighborhood of S is $N[S]=N(S)\cup S$.A subset S of V is called a dominating (total dominating) set if every vertex in V–S(V) is adjacent to some vertex in S. A dominating (total dominating) set is minimal dominating (total dominating) set if no proper subset of

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S is a dominating (total dominating) set of G. The domination number γ (total domination number γ_t) is the minimum cardinality taken over all minimal dominating (total dominating) sets of G. A γ-set (γ_t –set) is any minimal dominating (total dominating) set with cardinality γ (γ_t).

A proper coloring of G is an assignment of colors to the vertices of G, such that adjacent vertices have different colors. The smallest number of colors for which there exists a proper coloring of G is called chromatic number of G and is denoted by $\chi(G)$.

A dominator coloring on graphs is a proper coloring of graphs with the extra property that every vertex in the graph dominates an entire color class.The smallest number of colors for which there exists a dominator coloring of G is called dominator chromatic number of G and is denoted by $\chi_d(G)$.This concept was introduced by Raluca Gera et al [1].

In this paper we introduce a new concept total dominator coloring in G. Also we χ_{td} (G) for some classes of graphs and a general bound of this new parameter.

2.Main Results.

Section 2.1. In this section we introduce a new concept, total dominator coloring on graphs.

Definition 2.1. A total dominator coloring (td-coloring) on graphs with minimum degree at least one is a proper coloring of graphs with the extra property that every vertex in the graph properly dominates an entire color class. The total dominator chromatic number (td-chromatic number) of G is defined as minimum number of colors needed in a total dominator coloring of G and is denoted by χ_{td} (G). This concept is illustrated by the following example.

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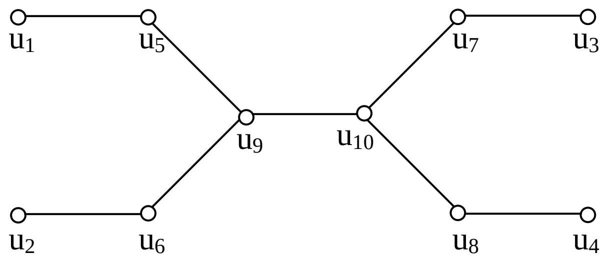

Figure1

In figure (1), G is a bipartite graph, so $\chi(G) = 2$. G has four end vertices u_1,u_2,u_3 and u_4 respectively, we assign color 1 to those vertices.We give the non-repeated colors 2 to 5 to the support vertices u_5,u_6,u_7 and u_8 respectively.The vertices $\{u_5,u_6\}$ and$\{u_7,u_8\}$ properly dominate the vertices u_9 and u_{10} respectively.So we assign color 6 and 7 to the vertices u_9 and u_{10} respectively.Thus each vertex in G properly dominates an entire color class. Thus G has required at most 7 colors for its td- coloring. Thus $\chi_{td}(G) \leq 7$.

Suppose the graph has $\chi_{td}(G) \leq 6$.The pendant vertex u_1 dominates only the vertex u_5.Thus u_5 is a color class. i.e., u_5 has a non-repeated color.Similarly u_6, u_7, u_8 all receive non-repeated colors.These four vertices receive non-repeated colors, say 1 to 4.Then the adjacent vertices u_9 and u_{10} receive two different colors say 5 and 6.Since the two vertices u_5 and u_6 have to dominate a color class,5 has to be a non-repeated color. Thus, the end vertices require one more color. Hence $\chi_{td}(G) = 7$.

Section2.2. In this section, we obtain the lower and upper bound for the newly introduced parameter and their characterization. Also we obtain a general bound.

The next two theorems give the characterization for lower and upper bounds of td-chromatic number of G.

Theorem 2.2. Let G be a connected graph of order p. Then $\chi_{td}(G)=2$ if and only if $G \cong K_{m,n}$ for some m, $n \in N$

Proof. First suppose $\chi_{td}(G) = 2$.Let C_1 and C_2 be the two color classes of G.Let $x \in C_1$.Since x can not dominates C_1,it should dominates C_2.Similarly for any vertex y∈C_2, y dominates C_1. Thus G is a complete bipartite graph with partition C_1 and C_2. Hence G ≅ $K_{m, n}$ for some m, n∈N.The converse is obvious.■

Theorem2.3. Let G be a connected graph of order p. Then χ_{td} (G) = p if and only if G ≅ K_p, for p ≥ 2.

Proof. Let G be a non-complete graph with $\delta(G) > 0$.We show that χ_{td} (G) < p. Let $u_1u_2 \notin E(G)$.We consider the following two cases. **Case (1).**δ(G) ≥ 2.We allot color 1 to u_1, u_2 and colors 2 to p–1 to the remaining p–2 vertices.This is clearly a td-coloring of G. We note that there will be a problem only if there is a vertex adjacent to u_1 or u_2 and to no other vertex.**Case(2).**δ(G)=1.Since G is non-complete, $p > 2$.We consider the following two sub cases.**Subcase (2.1).**G has at least two end vertices. We choose u_1 and u_2 to be two end vertices and proceed as in case (1), to show that χ_{td} (G) < p. **Subcase (2.2).**G has exactly one end vertex u_1 with support u_3.Let u_2 be a vertex adjacent to u_3.Proceeding as before, we show that χ_{td} (G) < p.The converse is obvious.■

Next we present a td-chromatic number for a disconnected graph with components $G_1, G_2, \ldots, G_k$, k ≥ 2.

Theorem 2.4. If G is a disconnected graph with non trivial components $G_1, G_2, ..., G_k$, k ≥ 2, then $\left(\max_{1 \le i \le k} \chi_{td}(G_i)\right) + 2k - 2 \le \chi_{td}(G) \le \sum_{i=1}^{k} \chi_{td}(G_i)$ and these bounds are sharp.

Proof. For each i (1 ≤ i ≤ k), the component G_i has color classes $C_{i_1}, C_{i_2}, ..., C_{i_{r_i}}$ Then $\bigcup_{i=1}^{k} C_{i1}, C_{i2}, ..., C_{iri}$ is a td- color class of G. Thus, χ_{td} (G) ≤ $\sum_{i=1}^{k} \chi_{td}(G_i)$.

Next we prove the lower bound. Let G_s be a component of G with maximum td-chromatic number.Then $\chi_{td}(G_s) = \max_{1 \le i \le k} \chi_{td}(G_i)$.For each i ≠ s, G_i needs at least

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two new colors, since each vertex in G_i properly dominates a color class.This establishes the required result.The lower and upper bound are sharp if $G \cong mK_2$.■

Remark 2.5. For any graph G, $\chi_{td}(G) \geq \omega(G)$, where $\omega(G)$ is the clique number of G.

Remark2.6. For every positive integer k, there is a (k + 2)-td-chromatic triangle free graph.

Next,we obtain a bound for td- chromatic number through the total domination number and chromatic number.

Theorem 2.7. Let G be any graph with $\delta(G) \geq 1$. Then max $\{\chi(G), \gamma_t(G)\} \leq \chi_{td}(G) \leq \chi(G) + \gamma_t(G)$.Also the bounds are sharp.

Proof. We have $\chi(G) \leq \chi_d(G) \leq \chi_{td}(G)$.Therefore, $\chi(G) \leq \chi_{td}(G)$.Let f be a minimal td-coloring of G. For each color class C_i ($1 \leq i \leq \chi_{td}(G)$), choose a vertex $x_i \in C_i$. Let $S=\{x_1, x_2, \ldots, x_{\chi td(G)}\}$. Now we have to show that S is a γ_t- set of G.Let $y \in$ G. Then y properly dominates a color class say C_i ($1 \leq i \leq \chi_{td}(G)$).In particular y is dominated by a vertex x_i.That is y dominates its open neighborhood N(y).Thus every vertex in G is adjacent to a vertex in S. Hence S is a γ_t- set of G.

To prove the upper bound, let g be a proper coloring of G with $\chi(G)$–colors.Now we assign colors $\chi(G) + 1, \chi(G) + 2, \ldots, \chi(G) + \gamma_t(G)$ to the vertices of a γ_t-set of G leaving the other vertices colored as before.This is a td- coloring of G since it is still a proper coloring and the total dominating set provides the color class that every vertex properly dominates.

$G = C_4$ equality holds for lower bound, and that the upper bound is sharp can be seen for P_p with p = 8 and $p \geq 12$, $p \neq 14,18$.■

Section 2.3. Total dominator coloring in join of two Graphs.

In this section,we prove that the total dominator colorings in join of two graphs are same as its proper coloring as well as its dominator coloring.

Theorem 2.8. For any connected graph $G=G_1+G_2$, $\chi_{td}(G)=\chi_d(G)=\chi(G)=\chi(G_1)+\chi(G_2)$.

Proof. Let $G=G_1+G_2$. We know that $\chi(G)=\chi(G_1)+\chi(G_2)$. Also any proper coloring of G is a td- coloring of G. Thus $\chi_{td}(G) = \chi_d(G)= \chi(G) = \chi(G_1) + \chi(G_2)$. ■

The following cases are the particular cases of the above theorem.

1. For a wheel graph W_p, $\chi_{td}(W_p) = \begin{cases} 4 & if\ p\ is\ even \\ 3 & if\ p\ is\ odd \end{cases}$

2. If a graph G has a vertex of full degree u, $\chi_{td}(G) = \chi_d(G) = \chi(G) = 1 + \chi(G - u)$.

3. If G is a complete k- partite graph $K_{m_1,m_2,\ldots,m_k}$, $\chi_{td}(K_{m_1,m_2,\ldots,m_k})$ = $\chi(\overline{K_{m_1}})+\chi(\overline{K_{m_2}})+\ldots+\chi(\overline{K_{m_k}})$ = k. ■

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