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**Total Dominator Colorings in Graphs**

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**Abstract**

A total dominator coloring of graphs with minimum degree at least one is a proper coloring of graphs with the extra property that every vertex in the graph properly dominates a color class. The total dominator chromatic number is denoted by \( \chi_{td}(G) \) and is defined by the minimum number of colors needed in a total dominator coloring of \( G \). In this paper we find \( \chi_{td}(G) \) for some classes of graphs and obtain a general bound. Also we obtain a characterization for lower and upper bound.

**Key words and Phrases:** Total domination number, chromatic number and total dominator chromatic number.

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1. **Introduction.**

All graphs considered in this paper are finite, undirected graphs and we follow standard definitions of graph theory as found in [3].

Let \( G=(V,E) \) be a graph of order \( p \) with minimum degree at least one. The open neighborhood \( N(v) \) of a vertex \( v \in V(G) \) consists of the set of all vertices adjacent to \( v \). The closed neighborhood of \( v \) is \( N[v]=N(v) \cup \{ v \} \). For a set \( S \subseteq V \), the open neighborhood \( N(S) \) is defined to be \( \bigcup_{v \in S} N(v) \), and the closed neighborhood of \( S \) is \( N[S]=N(S) \cup S \). A subset \( S \) of \( V \) is called a dominating (total dominating) set if every vertex in \( V-S(V) \) is adjacent to some vertex in \( S \). A dominating (total dominating) set is minimal dominating (total dominating) set if no proper subset of
S is a dominating (total dominating) set of G. The domination number \( \gamma \) (total domination number \( \gamma_t \)) is the minimum cardinality taken over all minimal dominating (total dominating) sets of G. A \( \gamma \)-set (\( \gamma_t \)-set) is any minimal dominating (total dominating) set with cardinality \( \gamma \) (\( \gamma_t \)).

A proper coloring of G is an assignment of colors to the vertices of G, such that adjacent vertices have different colors. The smallest number of colors for which there exists a proper coloring of G is called chromatic number of G and is denoted by \( \chi(G) \).

A dominator coloring on graphs is a proper coloring of graphs with the extra property that every vertex in the graph dominates an entire color class. The smallest number of colors for which there exists a dominator coloring of G is called dominator chromatic number of G and is denoted by \( \chi_d(G) \). This concept was introduced by Raluca Gera et al [1].

In this paper we introduce a new concept total dominator coloring in G. Also we \( \chi_{ad}(G) \) for some classes of graphs and a general bound of this new parameter.

2. Main Results.

Section 2.1. In this section we introduce a new concept, total dominator coloring on graphs.

Definition 2.1. A total dominator coloring (td-coloring) on graphs with minimum degree at least one is a proper coloring of graphs with the extra property that every vertex in the graph properly dominates an entire color class. The total dominator chromatic number (td-chromatic number) of G is defined as minimum number of colors needed in a total dominator coloring of G and is denoted by \( \chi_{ad}(G) \). This concept is illustrated by the following example.
In figure (1), G is a bipartite graph, so $\chi(G) = 2$. G has four end vertices $u_1, u_2, u_3$ and $u_4$ respectively, we assign color 1 to those vertices. We give the non-repeated colors 2 to 5 to the support vertices $u_5, u_6, u_7$ and $u_8$ respectively. The vertices \{u_5, u_6\} and \{u_7, u_8\} properly dominate the vertices $u_9$ and $u_{10}$ respectively. So we assign color 6 and 7 to the vertices $u_9$ and $u_{10}$ respectively. Thus each vertex in G properly dominates an entire color class. Thus G has required at most 7 colors for its td-coloring. Thus $td(G) \leq 7$.

Suppose the graph has $td(G) \leq 6$. The pendant vertex $u_4$ dominates only the vertex $u_5$. Thus $u_5$ is a color class. i.e., $u_5$ has a non-repeated color. Similarly $u_6, u_7, u_8$ all receive non-repeated colors. These four vertices receive non-repeated colors, say 1 to 4. Then the adjacent vertices $u_9$ and $u_{10}$ receive two different colors say 5 and 6. Since the two vertices $u_5$ and $u_6$ have to dominate a color class, 5 has to be a non-repeated color. Thus, the end vertices require one more color. Hence $td(G) = 7$.

Section 2.2. In this section, we obtain the lower and upper bound for the newly introduced parameter and their characterization. Also we obtain a general bound. The next two theorems give the characterization for lower and upper bounds of td-chromatic number of G.

**Theorem 2.2.** Let G be a connected graph of order p. Then $\chi_{td}(G) = 2$ if and only if $G \cong K_{m,n}$ for some $m, n \in \mathbb{N}$

**Proof.** First suppose $\chi_{td}(G) = 2$. Let $C_1$ and $C_2$ be the two color classes of G. Let $x \in C_1$. Since x can not dominates $C_1$, it should dominates $C_2$. Similarly for any vertex...
y ∈ C₂, y dominates C₁. Thus G is a complete bipartite graph with partition C₁ and C₂. Hence G ≅ Kₘ,ₙ for some m, n ∈ N. The converse is obvious. ■

**Theorem 2.3.** Let G be a connected graph of order p. Then χ_{td}(G) = p if and only if G ≅ Kₚ, for p ≥ 2.

**Proof.** Let G be a non-complete graph with δ(G) > 0. We show that χ_{td}(G) < p.

Let u₁, u₂ ∈ E(G). We consider the following two cases. **Case (1).** δ(G) ≥ 2. We allot color 1 to u₁, u₂ and colors 2 to p−1 to the remaining p−2 vertices. This is clearly a td-coloring of G. We note that there will be a problem only if there is a vertex adjacent to u₁ or u₂ and to no other vertex. **Case (2).** δ(G) = 1. Since G is non-complete, p > 2. We consider the following two sub cases. **Subcase (2.1).** G has at least two end vertices. We choose u₁ and u₂ to be two end vertices and proceed as in case (1), to show that χ_{td}(G) < p. **Subcase (2.2).** G has exactly one end vertex u₁ with support u₂. Let u₁ be a vertex adjacent to u₂. Proceeding as before, we show that χ_{td}(G) < p. The converse is obvious. ■

Next we present a td-chromatic number for a disconnected graph with components G₁, G₂, ..., Gₖ, k ≥ 2.

**Theorem 2.4.** If G is a disconnected graph with non trivial components G₁, G₂, ..., Gₖ, k ≥ 2, then
\[
\max_{1 \leq i \leq k} \chi_{td}(G_i) + 2k - 2 \leq \chi_{td}(G) \leq \sum_{i=1}^{k} \chi_{td}(G_i)
\]
and these bounds are sharp.

**Proof.** For each i (1 ≤ i ≤ k), the component Gᵢ has color classes Cᵢ₁, Cᵢ₂, ..., Cᵢᵣᵢ. Then
\[
\bigcup_{i=1}^{k} C_{i₁}, C_{i₂}, ..., C_{iᵣᵢ}
\]
is a td-color class of G. Thus, \( \chi_{td}(G) \leq \sum_{i=1}^{k} \chi_{td}(Gᵢ) \).

Next we prove the lower bound. Let Gₛ be a component of G with maximum td-chromatic number. Then \( \chi_{td}(Gₛ) = \max_{1 \leq i \leq k} \chi_{td}(Gᵢ) \). For each i ≠ s, Gᵢ needs at least
two new colors, since each vertex in \( G_i \) properly dominates a color class. This establishes the required result. The lower and upper bound are sharp if \( G \cong mK_2 \).

**Remark 2.5.** For any graph \( G \), \( \chi_{td}(G) \geq \omega(G) \), where \( \omega(G) \) is the clique number of \( G \).

**Remark 2.6.** For every positive integer \( k \), there is a \((k + 2)\)-\( td \)-chromatic triangle free graph.

Next, we obtain a bound for \( td \)-chromatic number through the total domination number and chromatic number.

**Theorem 2.7.** Let \( G \) be any graph with \( \delta(G) \geq 1 \). Then max \{ \( \chi(G) \), \( \gamma(G) \) \} \leq \( \chi_{td}(G) \) \leq \( \chi(G) + \gamma(G) \). Also the bounds are sharp.

**Proof.** We have \( \chi(G) \leq \chi_{td}(G) \leq \chi(G) \). Therefore, \( \chi(G) \leq \chi_{td}(G) \). Let \( f \) be a minimal \( td \)-coloring of \( G \). For each color class \( C_i (1 \leq i \leq \chi_{td}(G)) \), choose a vertex \( x_i \in C_i \). Let \( S = \{ x_1, x_2, \ldots, x_{\chi_{td}(G)} \} \). Now we have to show that \( S \) is a \( \gamma_t \)-set of \( G \). Let \( y \in G \). Then \( y \) properly dominates a color class say \( C_i (1 \leq i \leq \chi_{td}(G)) \). In particular \( y \) is dominated by a vertex \( x_i \). That is \( y \) dominates its open neighborhood \( N(y) \). Thus every vertex in \( G \) is adjacent to a vertex in \( S \). Hence \( S \) is a \( \gamma_t \)-set of \( G \).

To prove the upper bound, let \( g \) be a proper coloring of \( G \) with \( \chi(G) \)-colors. Now we assign colors \( \chi(G) + 1, \chi(G) + 2, \ldots, \chi(G) + \gamma(G) \) to the vertices of a \( \gamma_t \)-set of \( G \) leaving the other vertices colored as before. This is a \( td \)-coloring of \( G \) since it is still a proper coloring and the total dominating set provides the color class that every vertex properly dominates.

\( G = C_4 \) equality holds for lower bound, and that the upper bound is sharp can be seen for \( P_p \) with \( p = 8 \) and \( p \geq 12 \), \( p \neq 14, 18 \).

**Section 2.3. Total dominator coloring in join of two Graphs.**

In this section, we prove that the total dominator colorings in join of two graphs are same as its proper coloring as well as its dominator coloring.

**Theorem 2.8.** For any connected graph \( G = G_1 + G_2 \), \( \chi_{td}(G) = \chi_{td}(G_1) = \chi_{td}(G_2) = \chi(G_1) + \chi(G_2) \).
Proof. Let $G = G_1 + G_2$. We know that $\chi(G) = \chi(G_1) + \chi(G_2)$. Also any proper coloring of $G$ is a td-coloring of $G$. Thus $\chi_{td}(G) = \chi_{td}(G_1) = \chi(G) = \chi(G_1) + \chi(G_2)$. ■

The following cases are the particular cases of the above theorem.

1. For a wheel graph $W_p$, $\chi_{td}(W_p) = \begin{cases} 4 & \text{if } p \text{ is even} \\ 3 & \text{if } p \text{ is odd} \end{cases}$

2. If a graph $G$ has a vertex of full degree $u$, $\chi_{td}(G) = \chi_{td}(G) = \chi(G) = 1 + \chi(G - u)$.

3. If $G$ is a complete $k$-partite graph $K_{m_1, m_2, \ldots, m_k}$, $\chi_{td}(K_{m_1, m_2, \ldots, m_k}) = \chi(K_{m_1}) + \chi(K_{m_2}) + \ldots + \chi(K_{m_k}) = k$. ■

References


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