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Topological Properties on Measure space($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) with Measure condition

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Abstract:

A study of some topological properties on measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) with measure conditions, has been introduced to develop the extended versions of Heine-Borel property (HBP), countable compactness, Lindelof, finite intersection property, Bolzano-Weierstrass property. The inter-relationship of these extended properties are also studied.

Keywords:
e- Heine-Borel property,e- Countably compactness, e-Lindelof, e- Bolzano-Weierstrass property, measure space, σ – additivity, monotonocity and σ- sub-additivity property.

1. Introduction:

In this paper S. C. P. Halakatti has considered some properties of topological space and has introduced measure conditions on these topological properties. The author has also developed the extended versions of Heine-Borel Property (HBP), Lindelof, finite intersection property, countable compactness, Bolzano-Weierstrass property on a measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ), adopting general measure conditions like σ- additivity, σ- subadditivity, monotonicity. The extended version of these properties are redesignated versions of topological properties on ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ).This study has an interesting implications on the measure manifolds, the concept which was introduced by S. C. P. Halakatti [2]. It is interesting to know that the invariance of these properties under the measure invariant transformations from measure manifolds (M, $\mathcal{T}_1$, Σ_1, μ_1) onto ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) enriches the further study of analysis on the measure manifolds. The applications of such measure invariant properties on a measure manifold are in the field of Engineering science, Neuroscience and Brain Science. In our previous paper [2] we have introduced some basic definition on measure manifold.

Definition 1.1 Measureable Chart [2]

Let (U, ${}^{\mathcal{T}_1}/_{U}$, ${}^{\Sigma_1}/_{U}$) ⊆ (M, $\mathcal{T}_1$, Σ_1) be a non empty measurable subspace of (M, $\mathcal{T}_1$, Σ_1) if there exists a map, ϕ: (U, ${}^{\mathcal{T}_1}/_{U}$, ${}^{\Sigma_1}/_{U}$) → ϕ (U, ${}^{\mathcal{T}_1}/_{U}$, ${}^{\Sigma_1}/_{U}$) ⊆($\mathbb{R}^n$, $\mathcal{T}$, Σ) , satisfying the following conditions,

(i) ϕ if homeomorphism,
(ii) ϕ is measurable if for every measurable Vϵ ($\mathbb{R}^n$, $\mathcal{T}$, Σ), ϕ^{-1}(V) ϵ (M, $\mathcal{T}_1$, Σ_1).

Then ((U, ${}^{\mathcal{T}_1}/_{U}$, ${}^{\Sigma_1}/_{U}$), ϕ) is called a **measurable chart.**

Definition 1.2 Measure Chart [2]

A measurable chart ((U, ${}^{\mathcal{T}_1}/_{U}$, ${}^{\Sigma_1}/_{U}$), ϕ) equipped with a measure ${}^{\mu_1}/_{U}$ is called a measure chart, denoted by ((U, ${}^{\mathcal{T}_1}/_{U}$ ${}^{\Sigma_1}/_{U}$, ${}^{\mu_1}/_{U}$), ϕ) satisfying following condition,

(i) ϕ if homeomorphism,
(ii) ϕ is measurable function if for every measurable Vϵ ($\mathbb{R}^n$, $\mathcal{T}$, Σ,), ϕ^{-1}(V) ϵ (M, $\mathcal{T}_1$, Σ_1, μ_1) is measurable,
(iii) ϕ is measure invariant.

Then, the structure ((U,${}^{\mathcal{T}_1}/_{U}$, ${}^{\Sigma_1}/_{U}$, ${}^{\mu_1}/_{U}$),ϕ) is called a **measure chart**.

Definition 1.3 :- Measurable Atlas [2]

By an $\mathbb{R}^n$ measurable atlas of class C^k on M we mean a countable collection ($\mathbb{A}$, ${}^{\mathcal{T}_1}/_{\mathbb{A}}$, ${}^{\Sigma_1}/_{\mathbb{A}}$) of n-dimensional measurable charts ((U_i, ${}^{\mathcal{T}_1}/_{U_i}$, ${}^{\Sigma_1}/_{U_i}$), ϕ_i) for all i ϵ N on (M, $\mathcal{T}_1$, Σ_1) subject to the following conditions:

(a_1) $\bigcup_{i=1}^{\infty}\left(\left(U_i, {}^{\mathcal{T}_1}/_{U_i}, {}^{\Sigma_1}/_{U_i}\right), \varphi_i\right) = M$

i.e. the countable union of the measurable charts in ($\mathbb{A}$, ${}^{\Sigma_1}/_{\mathbb{A}}$, ${}^{\mathcal{T}_1}/_{\mathbb{A}}$) cover (M, $\mathcal{T}_1$, Σ_1)

(**a**2) For any pair of measurable charts $\left(\left(U_i, {}^{\mathcal{T}_1}/_{U_i}, {}^{\Sigma_1}/_{U_i}\right), \varphi_i\right)$ and $\left(\left(U_j, {}^{\mathcal{T}_1}/_{U_j}, {}^{\Sigma_1}/_{U_j}\right), \varphi_i\right)$ in $(\mathbb{A}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\Sigma_1}/_{\mathbb{A}})$, the transition maps $\phi_i \circ \varphi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are

(i) differentiable maps of class C^k $(K \geq 1)$

i.e. $\phi_i \circ \phi_j^{-1} : \phi_j (U_i \cap U_j) \rightarrow \phi_i (U_i \cap U_j) \subseteq (\mathbb{R}^n, \mathcal{T}, \Sigma)$

$\phi_j \circ \phi_i^{-1} : \phi_i (U_i \cap U_j) \rightarrow \phi_j (U_i \cap U_j) \subseteq (\mathbb{R}^n, \mathcal{T}, \Sigma)$

are differentiable maps of class C^k $(K \geq 1)$

(ii) measurable,

i.e. these two transition maps $\phi_i \circ \varphi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are measurable functions if,

(a) for any measurable subset $K \subseteq \phi_i (U_i \cap U_j)$, $(\phi_i \circ \phi_j^{-1})^{-1} (K) \in \phi_j (U_i \cap U_j)$ is also measurable,

(b) for any measurable subset $S \subseteq \phi_j (U_i \cap U_j)$, $(\phi_j \circ \phi_i^{-1})^{-1} (S) \in \phi_i (U_i \cap U_j)$ is also measurable.

□

Definition 1.4:- Restriction of Measure μ_1 on $(\mathbb{A}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\Sigma_1}/_{\mathbb{A}})$. [2]

Let $(M, \mathcal{T}_1, \Sigma_1, \mu_1)$ be a measure space and let $(\mathbb{A}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\Sigma_1}/_{\mathbb{A}}) \in (M, \mathcal{T}_1, \Sigma_1, {}_1)$ be a non-empty measureable Atlas. The measure ${}^{\mu_1}/_{\mathbb{A}}$ on $(\mathbb{A}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\Sigma_1}/_{\mathbb{A}})$ is called the **restriction of measure μ_1 on $(\mathbb{A}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\Sigma_1}/_{\mathbb{A}})$.**

Definition 1.5:- Measure Atlas [2]

The structure $(\mathbb{A}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}})$ is called measure Atlas if $(\mathbb{A}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\Sigma_1}/_{\mathbb{A}})$ is a measureable Atlas equipped with restricted measure ${}^{\mu_1}/_{\mathbb{A}}$.

□

Condition to be satisfied for Measure Atlas:-

Definition 1.7:- Measure Atlas [2]

By an $\mathbb{R}^n$ measure atlas of class C^k on M, we mean a countable collection $(\mathbb{A}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}})$ of n-dimensional measure charts $((U_i, {}^{\mathcal{T}_1}/_{U_i}, {}^{\Sigma_1}/_{U_i}, {}^{\mu_1}/_{U_i}), \phi_i)$ for all $i \in N$ on $(M, \mathcal{T}_1, \Sigma_1, \mu_1)$ satisfying the following conditions:-

(**a**1) $\bigcup_{i=1}^{\infty}\left(\left(U_i, {}^{\mathcal{T}_1}/_{U_i}, {}^{\Sigma_1}/_{U_i}, {}^{\mu_1}/_{U_i}\right), \varphi_i\right) = M$

i.e. the countable union of the measure charts in $(\mathbb{A}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}})$ cover $(M, \mathcal{T}_1, \Sigma_1, \mu_1)$.

(**a**2) For any pair of measure charts $\left(\left(U_i, {}^{\mathcal{T}_1}/_{U_i}, {}^{\Sigma_1}/_{U_i}, {}^{\mu_1}/_{U_i}\right), \varphi_i\right)$ and $\left(\left(U_j, {}^{\mathcal{T}_1}/_{U_j}, {}^{\Sigma_1}/_{U_j}, {}^{\mu_1}/_{U_j}\right), \varphi_j\right)$ in $(\mathbb{A}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\Sigma_1}/_{\mathbb{A}})$, the transition maps $\phi_i \circ \varphi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are

(i) differentiable maps of class C^k $(K \geq 1)$

i.e. $\phi_i \circ \phi_j^{-1} : \phi_j (U_i \cap U_j) \rightarrow \phi_i (U_i \cap U_j) \subseteq (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$,

$\phi_j \circ \phi_i^{-1} : \phi_i (U_i \cap U_j) \rightarrow \phi_j (U_i \cap U_j) \subseteq (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$

are differentiable maps of class C^k $(K \geq 1)$

(ii) measurable .

i.e. These two transition maps $\phi_i \circ \varphi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are measurable functions if

(a) for any measurable subset $K \subseteq \phi_i (U_i \cap U_j)$, $(\phi_i \circ \phi_j^{-1})^{-1} (K) \in \phi_j (U_i \cap U_j)$ is also measurable,

(b) for any measurable subset $S \subseteq \phi_j (U_i \cap U_j)$, $(\phi_j \circ \phi_i^{-1})^{-1} (S) \in \phi_i (U_i \cap U_j)$ is also measurable,

(**a**3) For any two measure atlases $(\mathbb{A}_1, {}^{\mathcal{T}_1}/_{\mathbb{A}_1}, {}^{\Sigma_1}/_{\mathbb{A}_1}, {}^{\mu_1}/_{\mathbb{A}_1})$ and $(\mathbb{A}_2, {}^{\mathcal{T}_1}/_{\mathbb{A}_2}, {}^{\Sigma_1}/_{\mathbb{A}_2}, {}^{\mu_1}/_{\mathbb{A}_2})$, we say that a mapping ,$T : \mathbb{A}_1 \rightarrow \mathbb{A}_2$ is measurable if $T^{-1}(E)$ is measurable for every measure chart $E = ((U, {}^{\mathcal{T}_1}/_{U}, {}^{\Sigma_1}/_{U}, {}^{\mu_1}/_{U}), \phi) \subset (\mathbb{A}_2, {}^{\mathcal{T}_1}/_{\mathbb{A}_2}, {}^{\Sigma_1}/_{\mathbb{A}_2}, {}^{\mu_1}/_{\mathbb{A}_2})$ and the mapping is measure preserving if ${}^{\mu_1}/_{\mathbb{A}_1} (T^{-1}(E)) = {}^{\mu_1}/_{\mathbb{A}_2} (E)$,where $\mathbb{A}_1 \sim \mathbb{A}_2$ and ${}^{\mu_1}/_{\mathbb{A}_1} = {}^{\mu_1}/_{\mathbb{A}_2}$.

Then we call T a transformation.

(**a**4) If a measurable transformation $T: \mathbb{A} \rightarrow \mathbb{A}$ preserves a measure μ_1, then we say that μ_1 is T-invariant (or invariant under T). If T is invertible and if both T and T^{-1} are measurable and measure preserving then we call T an invertible measure preserving transformation.

□

Two measure atlases $\mathbb{A}_1$ and $\mathbb{A}_2$ in $\mathbb{A}_m^k(M)$ are said to be **equivalent** if $(\mathbb{A}_1 \cup \mathbb{A}_2)$ in $\mathbb{A}_m^k(M)$.In order that $\mathbb{A}_1 \cup \mathbb{A}_2$ be a member of $\mathbb{A}_m^k(M)$ we require that for every measure chart $((U_i, {}^{\mathcal{T}_1}/_{U_i}, {}^{\Sigma_1}/_{U_i}, {}^{\mu_1}/_{U_i}), \phi_i) \in \mathbb{A}_1$ and every measure chart $(V_j, {}^{\mathcal{T}_1}/_{V_j}, {}^{\Sigma_1}/_{V_j}, {}^{\mu_1}/_{V_j}), \Psi_j) \in \mathbb{A}_2$

the set of $\phi_i(U_i \cap V_j)$ and $\Psi_j (U_i \cap V_j)$ be open measurable in $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ and maps $\phi_i \circ$

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$\Psi_j^{-1}$and ϕ_j o ϕ_i^{-1} be of class C^k and are measurable . The relation introduced is an equivalence relation in $\mathbb{A}_m^k(M)$ and hence partitions $\mathbb{A}_m^k(M)$ into disjoint equivalence classes . Each of these equivalence classes is called a **differentiable structure** of class C^k on (M, $\mathcal{T}_1$, Σ_1, μ_1). A measure space (M, $\mathcal{T}_1$, Σ_1, μ_1) together with a differentiable structure of class c^k is called a **differentiable measure** n- manifold of class C^k or simply a C^k –measure n-manifold .

A non empty set M equipped with differentiable structure, topological structure and algebraic structure σ- algebra is called **Measurable Manifold**. A measure μ_1 defined on (M, $\mathcal{T}_1$, Σ_1) and the quadruple (M, $\mathcal{T}_1$, Σ_1, μ_1) is called **Measure Manifold.**

Definition 1.7 Measurable Manifold[2]
A non-empty set M, which is modeled on measureable space ($\mathbb{R}^n$, $\mathcal{T}$, Σ) is called a **measurable manifold** denoted by (M, $\mathcal{T}_1$, Σ_1).

Definition 1.8 Measure Manifold [2]
If ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is a measure space a non-empty set M modelled on ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is called **Measure Manifold.**

Definition 1.9 Borel Cover [2].
By a **Borel cover** viz, $\{\bigcup_{i=1}^{\infty} A_i : A_i$'s are Borel sets$\}$, we mean a countable union of all Borel sets belonging to ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ).
In this paper The collection of Borel sets belonging to Σ are denoted by A,B,C…etc

1.11 Heine-Borel property (HBP) on Topological Space ($\mathbb{R}^n$,d)
For a subset A of the topological space ($\mathbb{R}^n$,d), A has the Heine- Borel property if every open covering of A admits a finite sub-covering. The Heine-Borel property on the measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) having two structures one topological and the other the algebraic structure, σ-algebra-Σ was introduced and proved in our paper [2]. Also the concept of Borel cover was introduced on measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ).

In this paper we consider the extended version of extended Heine-Borel property (EHBP). The advantage of this extended version of Heine-Borel property is that the ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) naturally admits e-countable compactness property and Extended version of Bolzano-Weierstrass property (eBWp) on ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ).

2. Preliminaries:
In this section we consider some basic definition on topological space ($\mathbb{R}^n$, d).

Definition 2.1 Local Basis[5][6][1]
Let ($\mathbb{R}^n$,d) be a topological space and let p ∈ ($\mathbb{R}^n$,d). A **local basis** at p is a collection $\mathfrak{B}_p$ of $\mathcal{T}$ such that,For every p ϵ ($\mathbb{R}^n$,d) there exists B ϵ $\mathfrak{B}_p$ and G ϵ $\mathcal{T}$ such that: p ϵ B ⊆ G.

Theorem 2.2 [5]
A family $\mathfrak{B}$ of subsets of a set $\mathbb{R}^n$ is a basis for some topology on ($\mathbb{R}^n$, d) if and only if:
(i). ($\mathbb{R}^n$, d) = ∪{B:B ϵ $\mathfrak{B}$} and
(ii). If $B_1, B_2 \in \mathfrak{B}$ and p ϵ $B_1 \cap B_2$ then there exists B ϵ $\mathfrak{B}$ such that q ϵ B and B ⊆ $B_1 \cap B_2$.

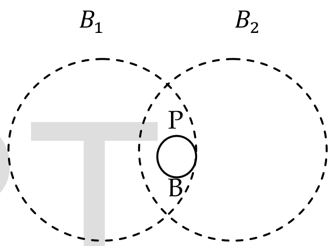

Fig1

Definition: 2.3 First Countable space[3][5]
A topological space ($\mathbb{R}^n$, d) is **first countable** provided there is countable local basis at each point of $\mathbb{R}^n$.

Definition: 2.4 Second Countable[5]
A topological space ($\mathbb{R}^n$, d) is **second countable** provided there is countable basis for $\mathcal{T}$.

Definition: 2.5 Compactness [3][5][1]
A subset A of a topological space ($\mathbb{R}^n$, d) is **compact** provided every open cover of A has a finite sub-cover.

Definition: 2.6 Countably Compact[4][6]
A topological space ($\mathbb{R}^n$, d) is **countably compact** provided every countable cover of $\mathbb{R}^n$ has a finite sub-cover.

Definition: 2.7 Locally Compact at a point[5]
A topological space ($\mathbb{R}^n$, d) is **locally compact at a point p** ∈ $\mathbb{R}^n$, provided there is an open set G and a compact subset K of ($\mathbb{R}^n$,d) such that p ∈ G ⊆ K.

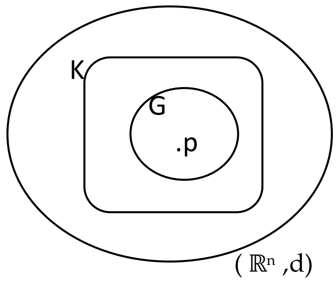

Fig 2

Definition 2.8 Locally Compact Topological Space ($\mathbb{R}^n$, d).
A topological space ($\mathbb{R}^n$, d) is **locally compact** if it is locally compact at each of its points.

Definition 2.9 Lindelof space[5][6]
A topological space ($\mathbb{R}^n$, d) is a **Lindelof space**, if every open cover has a countable subcover

Definition: 2.10 Heine- Borel property[HBP][5]
A subset A of a topological space ($\mathbb{R}^n$, d) has the **Heine Borel property,** if every open covering of A admits a finite sub covering.

Definition 2.11:- Finite Intersection Property (f.i.p)
A family $\mathbb{A}$ of subsets of a set has the **Finite intersection property**, if $\mathbb{A}'$ is a finite sub collection of then $\cap\{A: A \in \mathbb{A}'\} \neq \emptyset$.

Definition 2.12 Bolzano-Weierstrass property [5]
A topological space ($\mathbb{R}^n$,d) has the **Bolzano-Weierstrass property** provided that every infinite subset A of ($\mathbb{R}^n$,d) has a limit point.

Definition 2.13 A Measure on ($\mathbb{R}^n$, Σ, $\mathcal{T}$) [9]
Let ($\mathbb{R}^n$, Σ, $\mathcal{T}$) be a measure space. A function μ: Σ → [0, ∞] is called a **measure** on ($\mathbb{R}^n$, $\mathcal{T}$, Σ) or simply a measure on Σ if,

(i). $\mu(\emptyset) = 0$

(ii). $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ for all disjoint countable collection $\{A_i\}$ of measurable sets in Σ.

The topological space ($\mathbb{R}^n$,d) along with topological structure $\mathcal{T}$,an algebraic structure σ-algebra Σ, and a measure μ on ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is called a **measure space.**
Note that, the values of a measure are non-negative real numbers belonging to [0, ∞].

Property (ii) of a measure is called **σ-additivity** and sometimes a measure is also called **σ-additive measure**.

Theorem 2.14[9]
Let ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) be a measure space

(i). (Monotonicity) If A, B ∈ Σ and A ⊆ B then μ(A) ≤ μ(B)

(ii). If A, B ∈ Σ , A ⊆ B and μ(A) < ∞ ,then $\mu(B/A) = \mu(B) - \mu(A)$.

(iii). (σ- sub additivity) If $A_1, A_2, \ldots, \in \Sigma$ then $\mu(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \mu(A_i)$

Definition 2.15 [2]
Let ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) be a measure space

(i). μ is called **finite** if μ ($\mathbb{R}^n$) < ∞

(ii). μ is called **σ-finite** if there exists $A_1, A_2, \ldots, \in \Sigma$, so that $\mathbb{R}^n = \bigcup_{i=1}^{\infty} A_i$ and $\mu(A_i) < \infty$ for i ϵ N.

(iii) A set A ∈ Σ is called of σ-finite measure if there exists $A_1, A_2, \ldots, \in \Sigma$ so that

$A \subseteq \bigcup_{i=1}^{\infty} A_i$ and $\mu(A_i) < \infty$ for all i.

In Section 3, measure open cover means Borel cover.

3. Extended version of some Topological Properties on a Measure Space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ)

In this section Halakatti has developed the extended versions of Heine-Borel property, countable compactness, Bolzano-Weierstrass property on a measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ). It is observed that the measure space carries two structures, one the topological structure $\mathcal{T}$ and the other, the algebraic structure σ-algebra, Σ, on which a measure fuction μ has been introduced. The first author noticed that a few topological properties like Heine-Borel property, countable compactness, Bolzano- Finite intersection property (f.i.p), Weierstrass property on a usual topological space ($\mathbb{R}^n$,d), interestingly admits the extended versions of these topological properties on a newly structured space i.e., a measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) admiting σ-additivity, σ-subadditivity, monotonicity. Such an extension of topological property on ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) are re-designated as extended Heine-Borel property(eHBp), extended Bolzano-Weierstrass property(eBWp), extended finite intersection property (e.f.i.p),

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In this paper S.C.P. Halakatti has introduced and developed the extended versions of some basic definitions on measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ). Note that the following symbols are used in our definitions and theorems.

Note: 3.1

(1) The collection of open sets belonging to topological space ($\mathbb{R}^n$,d) are denoted by G.H…etc

(2) The collection of Borel sets belonging to the family subspace of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) are denoted by A,B,C…etc

(3) The members of cover for measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) are denoted by A_i,B_i,...,etc.

(4) Every measure open cover means Borel cover.

Definition 3.2 Local basis for measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ):-

Let ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) be a measure space and let q ∈ ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) then a **local basis** at a point q ϵ ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is a countable collection $\mathfrak{B}_q$ of a Borel sets ∈ Σ satisfying the following conditions:

(i). for each q ∈ ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ), there exist A ∈ Σ , B ∈ $\mathfrak{B}_q$ such that q ∈ B ⊆ A.

(ii). for B ⊆ A , μ (B) ≤ μ (A).... (Monotonicity)

($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ)

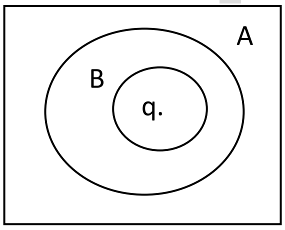


Fig3

Theorem 3.3

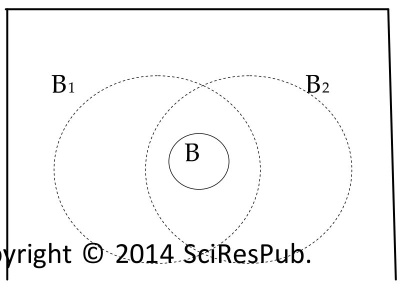
A family $\mathfrak{B}$ of Borel sets of a measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is a basis for ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) if it satisfies the following conditions:

(i). ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) = $\bigcup\{B: B \in \mathfrak{B}\}$ and

(ii). If B_1,B_2 ∈ $\mathfrak{B}$ and q ∈ $B_1 \cap B_2$ then, there exists B ∈ $\mathfrak{B}_q$ such that q ∈ B and B ⊆ $B_1 \cap B_2$., and

(iii). μ (B) ≤ μ ($B_1 \cap B_2$),

($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ)



q.

Fig 4

Proof:

(i) Since, ($\mathbb{R}^n$, $\mathcal{T}$) has a cover ∪ G_i **i.e.** ($\mathbb{R}^n$, $\mathcal{T}$) ⊆ ∪ G_i for each i ϵ I. Every open set G ϵ $\mathcal{T}$ generates a corresponding Borel set B ϵ Σ. Therefore there exists a open cover ∪{ B: B ϵ $\mathfrak{B}$ } i.e. Let $\mathfrak{B}$ be a family of Borel subsets of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) such that $\mathfrak{B}$ is a basis for ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ).

Therefore ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) = ∪{ B: B ϵ $\mathfrak{B}$ }.

(ii) Suppose B_1,B_2 ∈ $\mathfrak{B}$ and q ∈ $B_1 \cap B_2$ where $\mathfrak{B}$ is a basis for Σ, there is a sub collection $\mathfrak{B}'$ of $\mathfrak{B}$ such that , $B_1 \cap B_2 = \bigcup\{B: B \in \mathfrak{B}'\}$ satisfying σ-additivity , $\mu(B_1 \cap B_2) = \bigcup\{\mu(B): B \in \mathfrak{B}'\}$.

Therefore there exists B ϵ $\mathfrak{B}$ such that q ϵ B and B ⊆ $B_1 \cap B_2$, satisfying the σ –subadditivity property, μ (B) ≤ μ ($B_1 \cap B_2$), for q ϵ B and B ⊆ $B_1 \cap B_2$. □

Definition 3.4 First Countable Measure Space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ)(e-First Countable):

A measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is **first countable** if there is a collection $\mathfrak{B}_q$ of Borel sets as local base at each point q ϵ ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) satisfying the following conditions:

(i). for each q ∈ ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ), there exist $A \in \Sigma$, $B \in \mathfrak{B}_q$ such that q ∈ $B \subseteq A$.

(ii). for $B \subseteq A$ then μ (B) ≤ μ (A)...(Monotonicity)

A first countable measure space is redesignated as e- first countable.

Definition 3.5 Second Countable Measure Space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ)(e- Second Countable).

A measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is **second countable** provided there is a countable base for all q ∈ ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) satisfying the following conditions:

(i).for each q_i ϵ ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) there exist $B_i \in \mathfrak{B}_q$ and $A_i \in \Sigma$ such that $q_i \in B_i \subseteq A_i$ for i=1,2,3,….

(ii). for $B_i \subseteq A_i$, μ(B_i) ≤ μ(A_i)

A second countable measure space is redesignated as e- second countable

Definition 3.6

A collection ε of an arbitrary subset of a non-empty topological place $\mathbb{R}^n$ is said to generate σ-

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algebra $\Sigma(\varepsilon)$, if the intersection of all σ-algebra of subsets of $\mathbb{R}^n$ include ε, namely
$\Sigma(\varepsilon) = \cap \{ \Sigma : \Sigma$ is a σ-algebra of subsets of $\mathbb{R}^n$ and $\varepsilon \subseteq \Sigma \}$, the smallest σ-algebra.

Note that, there is at least one σ-algebra of subsets of $\mathbb{R}^n$, which includes ε and this is $\mathbb{P}(\mathbb{R}^n)$

Definition 3.7 Borel σ-algebra
Let $\mathbb{R}^n$ be a topological space and $\mathcal{T}$-the collection of all open subsets of $\mathbb{R}^n$ be a topology on $\mathbb{R}^n$. Then σ-algebra Σ generated by $\mathcal{T}$ containing all open subsets of $\mathbb{R}^n$, is called the Borel σ-algebra of $\mathbb{R}^n$, denoted by $\mathfrak{B}_{\mathbb{R}^n}$,
i.e $\mathfrak{B}_{\mathbb{R}^n} = \Sigma(\mathfrak{J})$.
The elements of $\mathfrak{B}_{\mathbb{R}^n}$ are Borel sets in $\mathbb{R}^n$.

Definition 3.8 Borel Cover [2].
By a **Borel cover** viz, $\{\cup_{i=1}^{\infty} A_i : A_i$'s are Borel sets$\}$, we mean a countable union of all Borel sets belonging to $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$.
In this paper The collection of Borel sets belonging to Σ are denoted by A,B,C…

Definition 3.9 Compact Measure Space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ)(e-Compact):

Let $(A, {}^{\mathcal{T}}/_{A}, {}^{\Sigma}/_{A}, {}^{\mu}/_{A}) \subseteq (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$, then $(A, {}^{\mathcal{T}}/_{A}, {}^{\Sigma}/_{A}, {}^{\mu}/_{A})$ is **compact measure space** if every Borel cover of pairwise disjoint sequence of measure sets i.e. $\cup_{i=1}^{\infty}(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i})$ of $(A, {}^{\mathcal{T}}/_{A}, {}^{\Sigma}/_{A}, {}^{\mu}/_{A})$ has a finite Borel sub cover, satisfying the following conditions:

(i) for every cover $\cup_{i=1}^{\infty}(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i})$ of a subset $(A, {}^{\mathcal{T}}/_{A}, {}^{\Sigma}/_{A}, {}^{\mu}/_{A})$ of $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ then there exists a finite sub cover which is also countable, i.e. $\cup_{j=1}^{n}(A_{i_j}, {}^{\mathcal{T}}/_{A_{i_j}}, {}^{\Sigma}/_{A_{i_j}}, {}^{\mu}/_{A_{i_j}})$, that covers $(A, {}^{\mathcal{T}}/_{A}, {}^{\Sigma}/_{A}, {}^{\mu}/_{A})$

(ii) $\mu\left(\cup_{i=1}^{\infty}(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i})\right) = \mu\left(\cup_{j=1}^{n}(A_{i_j}, {}^{\mathcal{T}}/_{A_{i_j}}, {}^{\Sigma}/_{A_{i_j}}, {}^{\mu}/_{A_{i_j}}\right)$
satisfying σ- subadditivity property,

$\sum_{j \subset I} \mu(A_{ij}, {}^{\Sigma}/_{A_{ij}}, {}^{\mathcal{T}}/_{A_{ij}}, {}^{\mu}/_{A_{ij}}) \leq \sum_{i=1}^{\infty} \mu(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i})$.

A compact measure space is redesignated as e-compact.

Definition 3.10 Countably Compact Measure Space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ)(e- Countably Compact)

A measure space $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ is **countably compact** if every countable Borel cover of $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ has a finite Borel sub cover, i.e.

(i) for every cover $\cup_{i=1}^{\infty}(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i})$ of a subset $(A, {}^{\mathcal{T}}/_{A}, {}^{\Sigma}/_{A}, {}^{\mu}/_{A})$ of $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$, then there exists a finite sub cover, i.e. $\cup_{j=1}^{n}(A_{i_j}, {}^{\mathcal{T}}/_{A_{i_j}}, {}^{\Sigma}/_{A_{i_j}}, {}^{\mu}/_{A_{i_j}})$, that covers $(A, {}^{\mathcal{T}}/_{A}, {}^{\Sigma}/_{A}, {}^{\mu}/_{A})$ of $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$, satisfying σ-subadditivity property,

(ii) $\mu\left(\cup_{i=1}^{\infty}(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i})\right) = \mu\left(\cup_{j=1}^{n}(A_{i_j}, {}^{\mathcal{T}}/_{A_{i_j}}, {}^{\Sigma}/_{A_{i_j}}, {}^{\mu}/_{A_{i_j}}\right)$
such that,
$\sum_{j \subset I} \mu(A_{ij}, {}^{\Sigma}/_{A_{ij}}, {}^{\mathcal{T}}/_{A_{ij}}, {}^{\mu}/_{A_{ij}}) \leq \sum_{i=1}^{\infty} \mu(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i})$.

A countable compact measure space is redesignated as e- countable compact.

Definition 3.11 Lindelof Measure Space(e-Lindelof)
A measure space $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ is a **Lindelof measure space** if, every Borel cover has a countable Borel subcover with σ subadditivity property, satisfying the following conditions:

(i). for every cover $\cup_{i=1}^{\infty}(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i})$ of a subset $(A, {}^{\mathcal{T}}/_{A}, {}^{\Sigma}/_{A}, {}^{\mu}/_{A})$ of $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ since finite sub cover is always countable sub cover, therefore, there exists a countable sub cover i.e.
$(\cup_{j=1}^{n}(A_{i_j}, {}^{\mathcal{T}}/_{A_{i_j}}, {}^{\Sigma}/_{A_{i_j}}, {}^{\mu}/_{A_{i_j}})$ that covers $(A, {}^{\mathcal{T}}/_{A}, {}^{\Sigma}/_{A}, {}^{\mu}/_{A})$

(ii). $\mu\left(\cup_{i=1}^{\infty}(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i})\right) = \mu\left(\cup_{j=1}^{n}(A_{i_j}, {}^{\mathcal{T}}/_{A_{i_j}}, {}^{\Sigma}/_{A_{i_j}}, {}^{\mu}/_{A_{i_j}}\right)$
satisfying σ- subadditivity property,

$\sum_{j \subset I} \mu(A_{ij}, {}^{\Sigma}/_{A_{ij}}, {}^{\mathcal{T}}/_{A_{ij}}, {}^{\mu}/_{A_{ij}}) \leq \sum_{i=1}^{\infty} \mu(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i})$.

A Lindelof measure space is redesignated as e-Lindelof.

Definition 3.12 Locally Compact at a point

of Measure Space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ)(e-locally compact)

A measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is **locally compact measure space** at a point q ∈ ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) if there is an Borel set B ∈ Σ and a compact subset (K ,$\mathcal{T}/_K$,$\Sigma/_K$, $\mu/_K$) of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) such that, (K ,$\mathcal{T}/_K$,$\Sigma/_K$, $\mu/_K$) ⊂ ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ)
($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ)

(i). For every q ∈ ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ), there exist

B ϵ Σ, K ⊂ $\mathcal{T}$ such that

q ϵ B ⊆ (K, $\mathcal{T}/_K$,$\Sigma/_K$, $\mu/_K$),

(ii). μ (B) ≤ μ (K)……… (Monotonocity)

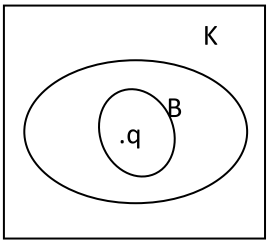


Fig 5

A measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is **locally compact** if it is locally compact at each of its points.

Definition 3.13 Extended Heine – Borel Property (eHBp)

Let (A, $\mathcal{T}/_A$,$\Sigma/_A$, $\mu/_A$) be a measure subspace of a measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) , then a measure subspace (A, $\mathcal{T}/_A$,$\Sigma/_A$, $\mu/_A$) has the extended Heine- Borel property if every open Borel cover of (A, $\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$) admits a finite Borel sub cover satisfying σ-sub additivity property:

(i) If (A,$\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$) ⊆ $\cup_{i \epsilon I}(A_i,\mathcal{T}/_{A_i} ,\Sigma/_{A_i} ,\mu/_{A_i}$) then there exists a finite sub cover for j ϵ J,
J ⊂I, such that (A,$\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$) ⊆ $\cup_{j=1}^{n}(A_{i_j},\mathcal{T}/_{A_{i_j}} ,\Sigma/_{A_{i_j}} ,\mu/_{A_{i_j}})$ satisfying,

(ii). μ (A,$\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$)
$= \mu\,(\cup_{j=1}^{n}(A_{i_j},\mathcal{T}/_{A_{i_j}} ,\Sigma/_{A_{i_j}} ,\mu/_{A_{i_j}}))$
$=\sum_{j=1}^{n} \mu\,(A_{i_j},\mathcal{T}/_{A_{i_j}} ,\Sigma/_{A_{i_j}} ,\mu/_{A_{i_j}})$, …
(σ-additivity).

A extended Heine- Borel property is redesignated as e- Heine- Borel property(eHBp)

Definition 3.14: Extended Finite Intersection Property(eFIP).

A family A of Borel subset of a measure space **(**$\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) has **the extended finite intersection property** if A' is a finite sub collection of A then ∩{ A, where A is countable family of Borel closed subset of **(**$\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) : A ϵ A' }≠ ∅, satisfying μ (A)>0.

A extended finite intersection property is redesignated as e-fip**.**

Definition 3.15 Extended Bolzano-Weierstrass Property [eBWp]:-

A measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) has the Extended Bolzano-Weierstrass property provided that every infinite subspace of measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) has a limit point, satisfying the following conditions:

(i). for any infinite subspace (A,$\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$) of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) there exist a limit point q ∈ (A,$\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$),

(ii). μ (A, $\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$) ≥ 0,

A extended Bolzano-Weierstrass property is redesignated as e- Bolzano-Weierstrass property(eBWp)

Theorem:-3.16

If Heine-Borel property (HBP) holds on topological space ($\mathbb{R}^n$, d) then, a measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) admits an extended Heine-Borel property (eHBp) satisfying

(i)) If (A,$\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$) ⊆ $\cup_{i \epsilon I}(A_i,\mathcal{T}/_{A_i} ,\Sigma/_{A_i} ,\mu/_{A_i}$) then there exists a finite sub cover for j ϵ J,
J ⊂I, such that (A,$\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$) ⊆ $\cup_{j=1}^{n}(A_{i_j},\mathcal{T}/_{A_{i_j}} ,\Sigma/_{A_{i_j}} ,\mu/_{A_{i_j}})$ satisfying,

(ii). μ (A,$\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$)
$= \mu(\cup_{j=1}^{n}(A_{i_j},\mathcal{T}/_{A_{i_j}} ,\Sigma/_{A_{i_j}} ,\mu/_{A_{i_j}}))$
$=\sum_{j=1}^{n} \mu\,(A_{i_j},\mathcal{T}/_{A_{i_j}} ,\Sigma/_{A_{i_j}} ,\mu/_{A_{i_j}})$
.(σ-additivity)

Alternatively, every countable Borel cover has a finite sub cover satisfying measure condition .

Proof:-

Suppose ($\mathbb{R}^n$, d)admits Heine-Borel property then every open covering of a subset A ⊆ ($\mathbb{R}^n$, d) admits a finite sub cover.

To show that a measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) admits an extended Heine-Borel property, it is necessary to observe that the σ- algebra structure on ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) allows a countable Borel open cover is made up of pairwise disjoint sequence of $\{(A_i, \mathcal{T}/_{A_i}, \Sigma/_{A_i}, \mu/_{A_i})\}$for a measure subset (A,$\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$) of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ). Hence to show that ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) admits an eHBp, it is sufficient to show that every countable Borel open cover has a finite Borel sub cover. Let us prove the conditions (i) and (ii) of the theorem.

(i) Let (A,$\mathcal{T}/_A$,$\Sigma/_A$,,$\mu/_A$) ⊆ ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is a measure sub space of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) and let $\{\cup_{i=1}^{\infty}(A_i, \mathcal{T}/_{A_i}, \Sigma/_{A_i}, \mu/_{A_i})\}$ be a countable Borel cover $(A_i, \mathcal{T}/_{A_i}, \Sigma/_{A_i}, \mu/_{A_i})$ which are pair wise disjoint sequence $\{(A_i, \mathcal{T}/_{A_i}, \Sigma/_{A_i}, \mu/_{A_i})\}$for a measure subset (A,$\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$).

i.e. (A, $\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$) ⊆ $\cup_{i=1}^{\infty}(A_i, \mathcal{T}/_{A_i}, \Sigma/_{A_i}, \mu/_{A_i})$ ……………………(1)

satisfying σ-additivity condition on measure,

μ (A,$\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$)
= μ ($\cup_{i=1}^{\infty}(A_i, \mathcal{T}/_{A_i}, \Sigma/_{A_i}, \mu/_{A_i})$)
= $\sum_{i=1}^{\infty}\mu(A_i, \mathcal{T}/_{A_i}, \Sigma/_{A_i}, \mu/_{A_i})$) ……(2)
(- additivity)

Since A ⊂ $\mathbb{R}^n$, by Heine-Borel property on ($\mathbb{R}^n$,d) it implies that every open cover has finite sub cover.viz { $\cup_{j=1}^{n} A_{i_j}$}, such that A⊂ $\cup_{j=1}^{n} A_{i_j}$………………………………(3)

Since {A_i: i ϵ I} are open sub sets in ($\mathbb{R}^n$, $\mathcal{T}$) and the open subsets of $\mathcal{T}$ generate Borel sets therefore the corresponding Borel cover $\{(A_i, \mathcal{T}/_{A_i}, \Sigma/_{A_i}, \mu/_{A_i})$: i ϵ I } is a countable open Borel cover for (A,$\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$) ⊂ ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ). Heine-Borel Property on ($\mathbb{R}^n$, d) implies, every open cover has finite sub-cover, correspondingly, every topological measure subspace having countable Borel cover has a finite Borel sub cover (A,$\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$) ⊂$\cup_{i=1}^{\infty}$ $(A_i, \mathcal{T}/_{A_i}, \Sigma/_{A_i}, \mu/_{A_i})$…………………..(4), satisfying the σ-subadditivity property of measure space,viz,

μ (A,$\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$)
= ($\cup_{i=1}^{\infty}$ $(A_i, \mathcal{T}/_{A_i}, \Sigma/_{A_i}, \mu/_{A_i})$)
=$\sum_{i=1}^{\infty}\mu(A_i, \mathcal{T}/_{A_i}, \Sigma/_{A_i}, \mu/_{A_i})$) ………(5)
(σ- additivity)

has a finite Borel sub cover, for j ϵ J, J ⊂ I

(A, $\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$) ⊂ $\cup_{j=1}^{n}(A_{i_j}, \mathcal{T}/_{A_{ij}}, \Sigma/_{A_{ij}}, \mu/_{A_{i_j}})$ ………………..(6)

Hence (1) is satisfied.

(2) Equation (6) satisfying -subadditivity condition of a measure, viz,

μ (A,$\mathcal{T}/_A$,$\Sigma/_A$,$\mu/_A$)
= ($\cup_{j=1}^{n}(A_{i_j}, \mathcal{T}/_{A_{ij}}, \Sigma/_{A_{ij}}, \mu/_{A_{i_j}}$: j ϵ J,J ⊂ I))
=$\sum_{j=1}^{n}\mu(A_{i_j}, \mathcal{T}/_{A_{ij}}, \Sigma/_{A_{ij}}, \mu/_{A_{i_j}})$,….(7)
(σ-additivity).

This implies that, every countable Borel cover has a finite Borel sub cover. Hence extended Heine-Borel property on ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) admits countable compactness on a measure space. □

Remark 3.17

(i) Heine-Borel Property on a topological space on ($\mathbb{R}^n$, $\mathcal{T}$) implies that every open cover has a finite sub cover. If the σ-algebraic structure is induced on a topological space ($\mathbb{R}^n$, $\mathcal{T}$) along with the measure μ , the σ-algebra structure transforms a topological space ($\mathbb{R}^n$, $\mathcal{T}$) into a topological measurable space and transforms open cover of ($\mathbb{R}^n$, $\mathcal{T}$) into a countable Borel cover of **(**$\mathbb{R}^n$, $\mathcal{T}$, Σ, μ). If this countable Borel open cover has a finite sub cover then a measure space **(**$\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) admits an extended version of Heine-Borel property satisfying σ-sub additivity.

(ii) The consequence of such extension of Heine-Borel property on **(**$\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is that the extended Heine-Borel property on ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) admits e-countable compactness on a measure space. Hence the above theorem can be restated as fallows.

Theorem 3.18

A measure space **(**$\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) admitting e-Heine-Borel property is e- countably compact satisfying σ-additivity property of a measure μ.

Proof:

The proof is obvious as in Theorem 3.16

Theorem 3.19

A measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) admitting extended Heine Borel property(eHBp) is e-Lindelof satisfying σ-subadditivity property.

Proof:

Suppose ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) admits eHBp. i.e every measure Borel covering made up of pairwise disjoint sequence of Borel sets viz, $\cup_{i\epsilon I}((A_i, \mathcal{T}/_{A_i}, \Sigma/_{A_i}, \mu/_{A_i})$ of a measure sub set

$(A, {}^{\mathcal{T}}/_{A}, {}^{\Sigma}/_{A}, {}^{\mu}/_{A})$ of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) has a finite measure Borel sub-covering, viz, $\cup_{j\in I}(A_{ij}, {}^{\Sigma}/_{A_{ij}}, {}^{\mathcal{T}}/_{A_{ij}}, {}^{\mu}/_{A_{ij}})$,

We now show that, every measure Borel cover has a countable measure sub cover. According to eHBp every Borel cover has a finite Borel sub cover , but every finite Borel sub cover is always countable sub cover,satisfying σ- sub additivety property. Therefore, for every Borel covering $\{\cup_{i\epsilon I}((A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i}))\}$ for a subset $(A, {}^{\mathcal{T}}/_{A}, {}^{\Sigma}/_{A}, {}^{\mu}/_{A})$ of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ), there exits a finite Borel sub cover, which also countable, i.e. $\{\cup_{j\epsilon J}((A_{ij}, {}^{\Sigma}/_{A_{ij}}, {}^{\mathcal{T}}/_{A_{ij}}, {}^{\mu}/_{A_{ij}})\}$ such that, when $(A, {}^{\mathcal{T}}/_{A}, {}^{\Sigma}/_{A}, {}^{\mu}/_{A}) \subseteq \cup_{i\epsilon I}(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i})$ then, $(A, {}^{\mathcal{T}}/_{A}, {}^{\Sigma}/_{A}, {}^{\mu}/_{A}) \subseteq \cup_{j\in J}(A_{ij}, {}^{\Sigma}/_{A_{ij}}, {}^{\mathcal{T}}/_{A_{ij}}, {}^{\mu}/_{A_{ij}})$ for J⊆I, satisfying the σ- subadditive property of measure i.e.

$\mu(\bigcup_{i\epsilon 1}^{\infty}(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i}))=$
$\mu(\cup_{j\subset I}(A_{ij}, {}^{\Sigma}/_{A_{ij}}, {}^{\mathcal{T}}/_{A_{ij}}, {}^{\mu}/_{A_{ij}}))$
for which,
$\sum_{j\subset I}\mu(A_{ij}, {}^{\Sigma}/_{A_{ij}}, {}^{\mathcal{T}}/_{A_{ij}}, {}^{\mu}/_{A_{ij}}) \leq$
$\sum_{i=1}^{\infty}\mu(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i})$.

This proves, every Borel cover has a countable Borel sub cover satisfying the σ- subadditive property of measure.

Hence eHBp implies e-Lindelof on a measure space □

Theorem 3.20

A measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is e-countably compact if and only if every countable family of Borel closed subsets of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) with the e-finite intersection property has a non-empty intersection,

i.e.

(i) if A′ is a finite sub collection of A, where A is countable family of Borel closed subset of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) then ∩{ A : A ϵ A′ }≠ ∅,

(ii) satisfying the measure condition μ (A)>0.

Proof:

Suppose ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is e-countable compact Let A = $\{(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i}) : i \epsilon I\}$ be a countable family of Borel closed subsets of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) with the extended finite intersection property

(i) Suppose, $\bigcap_{i=1}^{\infty}\{(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i}) = \emptyset$

Let $\mathfrak{A}$ = { ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) - $(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i})$: i ϵ I } is collection of Borel open subsets of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ). Then

Since $\bigcup_{i=1}^{\infty}\{(\mathbb{R}^n, \mathcal{T}, \Sigma\, \mu) - \left(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i}\right)\} =$
($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ)- $\bigcap_{i=1}^{\infty}\{(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i})\}$
= ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ)- ∅
= ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ)

$\mathfrak{A}$ is an Borel cover of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ), since ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is e-countably compact there exist a finite numbers i_1 , i_2 , i_3 ,..., i_n of I such that, $\cup_{j=1}\{(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) - (A_{ij}, {}^{\mathcal{T}}/_{A_{ij}}, {}^{\Sigma}/_{A_{ij}}, {}^{\mu}/_{A_{ij}}) : i\epsilon I, j = 1,2, \ldots n\}$ covers ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ).

Thus ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) = $\bigcup_{j=1}^{n}(\{(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) - (A_{ij}, {}^{\mathcal{T}}/_{A_{ij}}, {}^{\Sigma}/_{A_{ij}}, {}^{\mu}/_{A_{ij}})\}$
= ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) – $\bigcap_{j=1}^{n}\{(A_{ij}, {}^{\mathcal{T}}/_{A_{ij}}, {}^{\Sigma}/_{A_{ij}}, {}^{\mu}/_{A_{ij}})\}$

This implies $\bigcap_{j=1}^{n}\{(A_{ij}, {}^{\mathcal{T}}/_{A_{ij}}, {}^{\Sigma}/_{A_{ij}}, {}^{\mu}/_{A_{ij}})\} = \emptyset$ which is contradiction hence $\cup_{i\,\epsilon\, I}(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i}) \neq \emptyset$ satisfying μ measure condition $\mu\,(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i}) > 0$
$\mu\,\{\bigcap_{i=0}^{\infty}\{(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i})\} \neq \mu\,(\emptyset)$, implies $\mu\,(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i}) > 0$

(ii) Conversely,

Suppose every countable family of closed Borel sub sets of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) with the e-finite intersection property has a non empty intersection.

Let $\mathfrak{A}$ = $\{(B_i, {}^{\mathcal{T}}/_{B_i}, {}^{\Sigma}/_{B_i}, {}^{\mu}/_{B_i}) : i \epsilon I\}$ be a countable Borel cover of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ).

Suppose $\mathfrak{A}$ does not have a finite Borel sub-cover.

Let A = $\{(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) - (B_i, {}^{\mathcal{T}}/_{B_i}, {}^{\Sigma}/_{B_i}, {}^{\mu}/_{B_i}) : i \epsilon I\}$,

Then A is a countable family of Borel closed subsets of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ). Let J be a finite subset of I. Since $\mathfrak{A}$ does not have a finite Borel sub cover.

$\bigcap_{i\epsilon J}^{n}\{(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) - (B_i, {}^{\mathcal{T}}/_{B_i}, {}^{\Sigma}/_{B_i}, {}^{\mu}/_{B_i}) = (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) - \bigcup_{i\epsilon J}^{n}\{(B_i, {}^{\mathcal{T}}/_{B_i}, {}^{\Sigma}/_{B_i}, {}^{\mu}/_{B_i})\} \neq \emptyset$

Therefore A has the extended finite intersection property with $\mu\,(A_i, {}^{\mathcal{T}}/_{A_i}, {}^{\Sigma}/_{A_i}, {}^{\mu}/_{A_i}) > 0$. Hence $\bigcap_{i=1}^{\infty}\{(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) - (B_i, {}^{\mathcal{T}}/_{B_i}, {}^{\Sigma}/_{B_i}, {}^{\mu}/_{B_i})\} \neq \emptyset$

This is contradiction because $\bigcap_{i\epsilon I}\{(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) - (B_i, {}^{\mathcal{T}}/_{B_i}, {}^{\Sigma}/_{B_i}, {}^{\mu}/_{B_i})\} = (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) - \bigcup_{i\epsilon I}\{(B_i, {}^{\mathcal{T}}/_{B_i}, {}^{\Sigma}/_{B_i}, {}^{\mu}/_{B_i}) = \emptyset$

Therefore ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is e-countably compact

A measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is e-countably compact iff it is e- finite intersection property

□

Theorem 3.21

A measure space $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ admitting e-countably compact implicitly satisfies the e-Bolzano-Weierstrass property, with the following conditions,

(i) for any infinite subset $(A, \mathcal{T}/_A, \Sigma/_A, \mu/_A)$ of $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ there exist a limit point $q \in (A, \mathcal{T}/_A, \Sigma/_A, \mu/_A)$ where, $\mu(A, \mathcal{T}/_A, \Sigma/_A, \mu/_A) \geq 0$.

Proof:

Let $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ be a e-countably compact. Let $(A, \mathcal{T}/_A, \Sigma/_A, \mu/_A)$ be an infinite subset of $\mathbb{R}^n$ and let $(A, \mathcal{T}/_A, \Sigma/_A, \mu/_A)$ contains a countably infinite set of $B = \{q_i : i \in N\}$ we may assume that $i \neq j$ then $q_i \neq q_j$. The proof is by contradiction. Suppose B has no limit point and for each $n \in N$,

$C_n = \{q_i \in B : i \geq n\}$ is a Borel closed set in $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ satisfying the e-finite intersection property (e.f.i.p),

$\bigcap_{i=1}^{\infty}(C_n, \mathcal{T}/_{C_n}, \Sigma/_{C_n}, \mu/_{C_n}) \neq \emptyset$……. (1)

With measure condition,

$\mu(\bigcap_{i=1}^{\infty}(C_n, \mathcal{T}/_{C_n}, \Sigma/_{C_n}, \mu/_{C_n}) > 0$……….. (2)

But if $q_k \in B$, then q_k does not belongs to C_{k+1} and hence q_k does not belongs to $\bigcap_{n=1}^{\infty}(C_n, \mathcal{T}/_{C_n}, \Sigma/_{C_n}, \mu/_{C_n})$.

Therefore,

$$\bigcap_{n=1}^{\infty}(C_n, \mathcal{T}/_{C_n}, \Sigma/_{C_n}, \mu/_{C_n}) = \emptyset \text{ with}$$

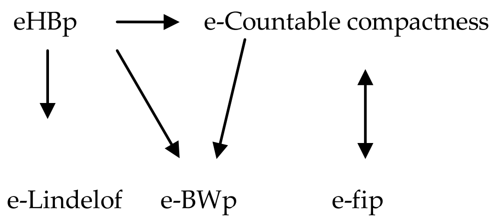
satisfies condition of measure is such that

$$\mu\left(\bigcap_{n=1}^{\infty}(C_n, \mathcal{T}/_{C_n}, \Sigma/_{C_n}, \mu/_{C_n})\right) = \mu(\emptyset) = 0$$

and we have a contradiction to our assumption. For $B = (B, \mathcal{T}/_B, \Sigma/_B, \mu/_B)$, $A = (A, \mathcal{T}/_A, \Sigma/_A, \mu/_A)$ Since B has a limit point q and $B \subseteq A$ with σ-additivity $\mu(B) \leq \mu(A)$, i.e. $q \in B \subseteq A$ such that $\mu(B) \leq \mu(A)$, therefore, $(A, \mathcal{T}/_A, \Sigma/_A, \mu/_A)$ has a limit point in $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$.

□

(1) A study on extended versions of some topological properties on a measure space $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ like e-Heine-Borel property, e-countable compactness , e-Lindelof, e- finite intersection property and e- Bolzano- Weierstrass property has shown that there exist a relationship between different forms of extended versions of compactness on a measure space $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ that is depicted below,



(2) One can observe that different topological properties of compactness and their interrelationship on a measure space $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ remains invariant under homeomorphism with respect to topological structure on $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$. In our future work, we will show that these topological properties when defined on measure space $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$, they may acquire measure conditions and remain invariant under measure invariant transformation. This approach has wider applications in building mathematical models for the problem in the field of Engineering science and Technology, also in the field of Brain science and Neuroscience.

Conclusion:-

One can notice that some topological properties can have a well defined measure on a measurable space $(\mathbb{R}^n, \mathcal{T}, \Sigma)$. When a non empty set M is modelled on such a measure space $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$, a measure manifold is generated adopting above extended properties.This is relevant in verifying the invariance of these properties under measure invariant transformation on the measure manifold.

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