International Journal of Advancements in Research & Technology, Volume 1, Issue 3, August-2012
ISSN 2278-7763

Study of Polarized Radiative Transfer Equation in a Coupled Atmosphere Ocean System with Reflecting and Transmitting Interface Boundaries using New Discrete Ordinate Method

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ABSTRACT

We have developed and analyzed coupled atmosphere-ocean polarized radiative transfer equations (RTE) system having flat boundary conditions at the interface between the two media. The boundary conditions depend on both reflection and transmission [1]. A study for analytical expressions for exit radiative intensities at any optical depth in the two media have been carried out using new version of discrete ordinate method (DOM) of Chandrasekhar first suggested by Siewert [2]. Each homogeneous version of equations is replaced by N equivalent directions, with corresponding Gaussian weight functions. Finally the Gaussian summation is interchanged with Fourier summation. The elementary solutions developed to accommodate for the inhomogeneous source term is used to construct the Green's function which are required to find the particular solution.

KEYWORDS: Coupled atmosphere ocean system; Discrete Ordinate Method; Discrete ordinate equations; Radiative transfer equation

1. INTRODUCTION

We use the elementary solutions of the homogeneous DOE [3 & 4] to develop the infinite medium Green's function, which is then used to construct a particular solution of a general form of the inhomogeneous DOE. The atmosphere ocean system is both absorbing and scattering and the extinction coefficient, single scattering coefficient albedo. The +Z-axis is taken downwards such as Z=0 and Z_W corresponds to the top of the atmosphere and ocean respectively, where Z_1 represents the lower limit (surface) of the ocean. The modeling of radiative transfer depends on reflection and refraction both at the interfaces [5] between the two media. The plane parallel model for both the media will be adopted so that the properties depend only on the vertical co-ordinates. The Stokes vector $I(Z,\mu,\phi)$ at each point of the medium is assumed to be a continuous function of μ,ϕ where μ is the absolute value of cosine of Θ, the elevation angle with the +Z and ϕ is the azimuthal angle measured relative to the +X-axis. We shall denote $I_{AT}(Z,\mu,\phi)$ for atmosphere and $I_{OC}(Z,\mu,\phi)$ for ocean medium. The atmosphere is illuminated from above only by sunlight incident at an angle μ_0 to the radiation and the only solar flux crossing a plane perpendicular to the direction of incidence is πF. Equations of transfer for non polarized and polarized radiation field along with the boundary conditions for flat ocean surface and the bottom of the ocean with the help of reflection and transmission have been worked upon. A new version of

International Journal of Advancements in Research & Technology, Volume 1, Issue 3, August-2012
ISSN 2278-7763

Chandrasekhar's DOM is used to solve the RTE. The homogeneous version of reduced coupled non linear integro differential RTE for atmosphere and ocean is used for each Stokes vector for discretization. We use Chandrasekhar's discretization scheme to break this continuous radiation field into 2N quadrature directions with corresponding weights keeping optical depth dependence exact. This enables to write N equations for each I (negative as well as positive). Each homogeneous version of equations is replaced by N equivalent directions, with corresponding Gaussian weight functions expressed and interchanging the Gaussian summation with Fourier summation. Computation of eigen functions and eigen values are worked in detail along with wind dependent albedo which finds a prominent place in further discussions. The elementary solutions developed previously by C.E.Siewert to accommodate for the inhomogeneous source term is used to construct the Green's functions which are required to find the particular solution. The four boundary conditions are used to find the unknown constants. Finally the homogeneous solution added together with the particular solution gives the intensity for any optical depth in both the media. We have meticulously found results of the four outgoing Stokes parameter from ocean for different directions.

2. THEORY

2.1 Equation of Polarized Radiative Transfer

The polarized radiative transfer equation for atmosphere and ocean are given in (1) and (2)

$$\mu\frac{dI_{AT}(z,\mu,\varphi)}{dz} = -I_{AT}(z,\mu,\varphi) + \frac{\omega^{AT}(z)}{4\pi}\int_{0}^{2\pi} d\varphi' + \int_{-1}^{+1} P_{AT}(\Theta)\, I_{AT}(z,\mu',\varphi')d\mu' + S_{AT}(z,\mu,\varphi). \quad (1)$$

$$\mu\frac{dI_{OC}(z,\mu,\varphi)}{dz} = -I_{OC}(z,\mu,\varphi) + \frac{\omega^{OC}(z)}{4\pi}\int_{0}^{2\pi} d\varphi' + \int_{-}^{+} P_{OC}(\Phi)\, I_{OC}(z,\mu',\varphi')d\mu' + S_{OC}(z,\mu,\varphi). \quad (2)$$

The general structure of source function in the above equations for both atmosphere and ocean are expressed as

$$S_{AT}(z,\mu,\varphi) = \frac{\omega^{AT}(z)}{4\pi}F_{0,1}P_{AT}(z,\mu,-\mu_0,\varphi_0)\exp\left(\frac{-z}{\mu_0}\right) + \frac{\omega^{AT}(z)}{4\pi}F_{0,1}R_{AT}(-\mu_0,n)P_{AT}(z,\mu,\varphi,\mu_0,\varphi_0)\exp\left(\frac{-(2z_\omega - z)}{\mu_0}\right). \quad (3)$$

$$S_{OC}(z,\mu,\varphi) = \frac{\omega^{OC}(z)}{4\pi}\frac{\mu_0}{\mu_{0n}(\mu_0,n)}F_{0,2}T_{OC}(-\mu_0,n) \times P_{OC}(z,\mu,\varphi,-\mu_{0n},\varphi_0)\exp\left(-\frac{z_\omega}{\mu_0}\right)\times\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right). \quad (4)$$

Where $F_{0,1}$ and $F_{0,2}$ are solar irradiances at top of the atmosphere and at the top of ocean respectively.

μ_0 = Cosine of the solar zenith angle and is positive. φ_0 =Azimuthal angle for incident solar beam.

n =Index of refraction of the ocean relative to the atmosphere. z_ω =Optical depth of the atmosphere. μ_{0n} = Cosine of the solar zenith angle in the ocean which is related to μ_0 by the Snell Law

$$\mu_{0n}(\mu_0,n) = \sqrt{1-\frac{(1-\mu_0^2)}{n^2}}. \quad (5)$$

The general structure of intensities in both atmosphere and ocean has been written as four stokes vector given as under

$$I_{AT/OC}(z,\mu,\varphi)=\begin{bmatrix} L_{AT/OC}(z,\mu,\varphi) \\ Q_{AT/OC}(z,\mu,\varphi) \\ U_{AT/OC}(z,\mu,\varphi) \\ V_{AT/OC}(z,\mu,\varphi) \end{bmatrix}. \quad (6)$$

The general structure of phase functions for both the mediums in equations (1) and (2) are given by the following expressions $$P_{AT/OC}(z,\mu,\mu',\varphi,\varphi')=L(-\chi)P_{AT/OC}(\cos\Theta_{AT}/\cos\Theta_{OC})L(\chi').$$
(7)

Where $L(\chi)$ denote the rotation matrix

$$L(\chi)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\chi & \sin 2\chi & 0 \\ 0 & -\sin 2\chi & \cos 2\chi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$
(8)

The boundary conditions for the above problem at the top of the atmosphere, interface at atmosphere and ocean and finally at the bottom of the ocean have been mentioned below.

$$I_{AT}(0,-\mu,\varphi)=I_{AT\infty}(-\mu,\varphi). \quad (9)$$

$$I_{AT}(z_\omega,\mu)=R_{AT}(-\mu,n)I_{AT}(z_\omega,-\mu)+T_{OC}(+\mu,n)\left\{\frac{I_{OC}(z_\omega,\mu)}{n^2}\right\}. \quad (10)$$

$$\frac{I_{OC}(z_\omega,-\mu)}{n^2}=R_{OC}(+\mu,n)\left\{\frac{I_{OC}(z_\omega,\mu)}{n^2}\right\}+T_{AT}(-\mu,n)I_{AT}(z_\omega,-\mu). \quad (11)$$

$$I_{OC}(z,\mu)=I_{OC}\cdot g(\mu).$$
(12)

R and **T** is the reflection and transmission matrix, given by the expressions

$$R=\frac{1}{2}\begin{bmatrix} r_L r_L^* + r_R r_R^* & r_L r_L^* - r_R r_R^* & 0 & 0 \\ r_L r_L^* - r_R r_R^* & r_L r_L^* + r_R r_R^* & 0 & 0 \\ 0 & 0 & r_L r_R^* + r_R r_L^* & r_L r_R^* - r_R r_L^* \\ 0 & 0 & r_L r_R^* - r_R r_L^* & r_L r_R^* + r_R r_L^* \end{bmatrix}.$$
(13)

$$T=\frac{1}{2}\begin{bmatrix} t_L t_L^* + t_R t_R^* & t_L t_L^* - t_R t_R^* & 0 & 0 \\ t_L t_L^* - t_R t_R^* & t_L t_L^* + t_R t_R^* & 0 & 0 \\ 0 & 0 & t_L t_R^* + t_R t_L^* & t_L t_R^* - t_R t_L^* \\ 0 & 0 & -t_L t_R^* + t_R t_L^* & t_L r_R^* + t_L t_R^* \end{bmatrix}. .$$
(14)

Where,

$$r_L = \frac{\sqrt{m^2 - \sin^2\mu} - m^2\cos\mu}{\sqrt{m^2 - \sin^2\mu} + m^2\cos\mu}, \quad (15) \qquad r_R = \frac{\cos\mu - \sqrt{m^2 - \sin^2\mu}}{\cos\mu + \sqrt{m^2 - \sin^2\mu}},$$

(16)

$$t_L = \frac{2\sqrt{m^2 - \sin^2\mu}}{\sqrt{m^2 - \sin^2\mu} + m^2\cos\mu} \quad (17) \qquad t_R = \frac{2\sqrt{m^2 - \sin^2\mu}}{\cos\mu + \sqrt{m^2 - \sin^2\mu}}$$

(18)

We have decomposed the intensity vector as

$$I_{AT}(z,\mu,\varphi) = \begin{bmatrix} L_{AT}(z,\mu,\varphi) \\ Q_{AT}(z,\mu,\varphi) \\ U_{AT}(z,\mu,\varphi) \\ V_{AT}(z,\mu,\varphi) \end{bmatrix} = \sum_{S=0}^{L}(2-\delta_{0,S}) \begin{bmatrix} L^S_{AT}(z,\mu)\cos S(\varphi-\varphi_S) \\ Q^S_{AT}(z,\mu)\cos S(\varphi-\varphi_S) \\ U^S_{AT}(z,\mu)\sin S(\varphi-\varphi_S) \\ V^S_{AT}(z,\mu)\sin S(\varphi-\varphi_S) \end{bmatrix}. \quad (19)$$

$$I_{OC}(z,\mu,\varphi) = \begin{bmatrix} L_{OC}(z,\mu,\varphi) \\ Q_{OC}(z,\mu,\varphi) \\ U_{OC}(z,\mu,\varphi) \\ V_{OC}(z,\mu,\varphi) \end{bmatrix} = \sum_{S=0}^{L}(2-\delta_{0,S}) \begin{bmatrix} L^S_{OC}(z,\mu) & \cos S(\varphi-\varphi_S) \\ Q^S_{OC}(z,\mu) & \cos S(\varphi-\varphi_S) \\ U^S_{OC}(z,\mu) & \sin S(\varphi-\varphi_S) \\ V^S_{OC}(z,\mu) & \sin S(\varphi-\varphi_S) \end{bmatrix}. \quad (20)$$

The Phase functions are decomposed as follows:

$$P_{AT/OC}(\mu,\varphi,\mu',\varphi') = \sum_{S=0}^{L}\frac{1}{2}(2-\delta_{0,S})[\cos s(\varphi-\varphi')P^S_C(\mu,\mu') + \sin s(\varphi-\varphi')P^S_S(\mu,\mu')]. \quad (21)$$

The immediate result of using these decompositions into the RTE (1) gives us after dropping S

$$\mu\frac{dI_{AT}(z,\mu)}{dz} = -I_{AT}(z,\mu) + \frac{\omega^{AT}(z)}{4}\int_{-1}^{+1}P_{AT}(z,\mu,\mu')\,I^S_{AT}(z,\mu')d\mu' + S^S_{AT}(z,\mu). \quad (22)$$

$$P_{AT}(z,\mu,\mu') = \sum_{J=S}^{M}\begin{bmatrix} P^S_J(\mu) & 0 & 0 & 0 \\ 0 & R^S_J(\mu) & -T^S_J(\mu) & 0 \\ 0 & -T^S_J(\mu) & R^S_J(\mu) & 0 \\ 0 & 0 & 0 & P^S_J(\mu) \end{bmatrix}\begin{bmatrix} \beta_J & \gamma_J & 0 & 0 \\ \gamma_J & \alpha_J & 0 & 0 \\ 0 & 0 & \xi_J & -\varepsilon_J \\ 0 & 0 & \varepsilon_J & \delta_J \end{bmatrix}\begin{bmatrix} P^S_J(\mu') & 0 & 0 & 0 \\ 0 & R^S_J(\mu') & -T^S_J(\mu') & 0 \\ 0 & -T^S_J(\mu') & R^S_J(\mu') & 0 \\ 0 & 0 & 0 & P^S_J(\mu') \end{bmatrix}$$

$$= \quad \mathbf{P}_J^S(\mu)\Pi_J^{AT}\mathbf{P}_J^S(\mu') \tag{22a}$$

Where,

$$\mathbf{P}_J^S(\mu) = \begin{bmatrix} P_J^S(\mu) & 0 & 0 & 0 \\ 0 & R_J^S(\mu) & -T_J^S(\mu) & 0 \\ 0 & -T_J^S(\mu) & R_J^S(\mu) & 0 \\ 0 & 0 & 0 & P_J^S(\mu) \end{bmatrix} \tag{22b}$$

Writing the source function in (3) for atmosphere in compact notation as

$$S_{AT}^S(z,\mu) = \frac{\omega^{AT}(z)}{4}\exp\left(-\frac{z}{\mu_0}\right).[E_{AT}(z,\mu).F]. \tag{23}$$

Where the expression for $E_{AT}(z,\mu)$ can be deduced from

$$E_{AT}(z,\mu,\varphi) = \frac{\omega^{AT}(z)}{4\pi}\left[P_{AT}(z,\mu,\varphi,-\mu_0,\varphi_0)\exp\left(-\frac{z}{\mu_0}\right) + P(z,\mu,\varphi;\mu_0,\varphi_0)R(-\mu_0,n)\exp\left(\frac{-(2z_\omega - z)}{\mu_0}\right)\right]. \tag{24}$$

Similarly the RTE (2) after decomposition can be written as

$$\mu\frac{dI_{OC}^S(z,\mu,)}{dz} = -I_{OC}^S(z,\mu,) + \frac{\omega^{OC}(z)}{4}\int_{-1}^{+1} P_{OCS}(z,\mu,\mu')\,I_{OC}^S(z,\mu')d\mu' + S_{OC}^S(z,\mu) \tag{25}$$

with,

$$P_{OC}(z,\mu,\mu') = \sum_{J=S}^{M}\begin{bmatrix} P_J^S(\mu) & 0 & 0 & 0 \\ 0 & R_J^S(\mu) & -T_J^S(\mu) & 0 \\ 0 & -T_J^S(\mu) & R_J^S(\mu) & 0 \\ 0 & 0 & 0 & P_J^S(\mu) \end{bmatrix}\begin{bmatrix} \beta'_J & \gamma'_J & 0 & 0 \\ \gamma'_J & \alpha'_J & 0 & 0 \\ 0 & 0 & \xi'_J & -\varepsilon'_J \\ 0 & 0 & \varepsilon'_J & \delta'_J \end{bmatrix}\begin{bmatrix} P_J^S(\mu') & 0 & 0 & 0 \\ 0 & R_J^S(\mu') & -T_J^S(\mu') & 0 \\ 0 & -T_J^S(\mu') & R_J^S(\mu') & 0 \\ 0 & 0 & 0 & P_J^S(\mu') \end{bmatrix}$$

$$= \quad \mathbf{P}_J^S(\mu)\Pi_J^{OC}\mathbf{P}_J^S(\mu')$$

(25a)

Again writing the source function for ocean in more compact notation as

$$S_{OC}^S(z,\mu) = \frac{\omega^{OC}(z)}{4}\exp\left(-\frac{z}{\mu_0}\right).[E_{OC}(z,\mu).F]. \tag{26}$$

Similarly, the expression for $E_{OC}(z,\mu)$ can be deduced from

$$E_{OC}(z,\mu,\varphi)=\left[P_{OC}(z,\mu,\varphi,-\mu_{0n},\varphi_0).T(-\mu_0,n).\exp\left(\frac{-(z-z_\omega)}{\mu_{0n}}\right).\left(\frac{\mu_0}{\mu_n}\right).\exp\left(-\frac{z_\omega}{\mu_0}\right)\right]. \tag{27}$$

We have derived the components of source functions for each stokes components as follows.

$$\begin{aligned}S^S_{ATL}(z,\mu)=\frac{\omega^{AT}(z)}{4}\Bigg[&\sum_{J=S}^{M}(-1)^{J+S}\beta_J P^S_J(\mu)P^S_J(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]+\Bigg\{\frac{1}{2}(r_Lr_L^*+r_Rr_R^*)\sum_{J=S}^{M}\beta_J P^S_J(\mu)P^S_J(\mu_0)\\&+\frac{1}{2}(r_Lr_L^*-r_Rr_R^*)\sum_{J=S}^{M}(\gamma_J P^S_J(\mu)R^S_J(\mu_0)\Bigg\}\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\Bigg]F_1+\frac{\omega^{AT}(z)}{4}\Bigg[\sum_{J=S}^{M}(-1)^{J+S}\gamma_J P^S_J(\mu)R^S_J(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]\\&+\Bigg\{\frac{1}{2}(r_Lr_L^*-r_Rr_R^*)\sum_{J=S}^{M}\beta_J P^S_J(\mu)P^S_J(\mu_0)+\frac{1}{2}(r_Lr_L^*+r_Rr_R^*)\sum_{J=S}^{M}\gamma_J P^S_J(\mu)R^S_J(\mu_0)\Bigg\}\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\Bigg]F_2+0+0.\end{aligned} \tag{28}$$

$$\begin{aligned}S^S_{ATQ}(z,\mu)=\frac{\omega^{AT}(z)}{4}\Bigg[&\sum_{J=S}^{M}(-1)^{J+S}\gamma_J R^S_J(\mu)P^S_J(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]+\Bigg\{\frac{1}{2}(r_Lr_L^*+r_Rr_R^*)\sum_{J=S}^{M}\gamma_J R^S_J(\mu)P^S_J(\mu_0)\\&+\frac{1}{2}(r_Lr_L^*-r_Rr_R^*)\sum_{J=S}^{M}(\alpha_J R^S_J(\mu)R^S_J(\mu_0)+\xi_J T^S_J(\mu)T^S_J(\mu_0))\Bigg\}\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\Bigg]F_1+\\&\frac{\omega^{AT}(z)}{4}\Bigg[\sum_{J=S}^{M}(-1)^{J+S}(\alpha_J R^S_J(\mu)R^S_J(\mu_0)+\xi_J T^S_J(\mu)T^S_J(\mu_0))\exp\left[-\frac{z}{\mu_0}\right]+\Bigg\{\frac{1}{2}(r_Lr_L^*-r_Rr_R^*)\sum_{J=S}^{M}\gamma_J R^S_J(\mu)P^S_J(\mu_0)\\&+\frac{1}{2}(r_Lr_L^*+r_Rr_R^*)\sum_{J=S}^{M}(\alpha_J R^S_J(\mu)R^S_J(\mu_0)+\xi_J T^S_J(\mu)T^S_J(\mu_0))\Bigg\}\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\Bigg]F_2+0+0\,.\end{aligned} \tag{29}$$

$$\begin{aligned}S^S_{ATU}(z,\mu)=\frac{\omega^{AT}(z)}{4}\Bigg[&-\sum_{J=S}^{M}(-1)^{J+S}\gamma_J T^S_J(\mu)P^S_J(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]+\Bigg\{-\frac{1}{2}(r_Lr_L^*+r_Rr_R^*)\sum_{J=S}^{M}\gamma_J T^S_J(\mu)P^S_J(\mu_0)\\&-\frac{1}{2}(r_Lr_L^*-r_Rr_R^*)\sum_{J=S}^{M}(\alpha_J T^S_J(\mu)R^S_J(\mu_0)+\xi_J R^S_J(\mu)T^S_J(\mu_0))\Bigg\}\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\Bigg]F_1+\\&\frac{\omega^{AT}(z)}{4}\Bigg[-\sum_{J=S}^{M}(-1)^{J+S}(\alpha_J T^S_J(\mu)R^S_J(\mu_0)+\xi_J R^S_J(\mu)T^S_J(\mu_0))\exp\left[-\frac{z}{\mu_0}\right]+\Bigg\{-\frac{1}{2}(r_Lr_R^*-r_Rr_L^*)\sum_{J=S}^{M}\gamma_J T^S_J(\mu)P^S_J(\mu_0)\\&-\frac{1}{2}(r_Lr_L^*+r_Rr_R^*)\sum_{J=S}^{M}(\alpha_J T^S_J(\mu)R^S_J(\mu_0)+\xi_I R^S_J(\mu)T^S_J(\mu_0))\Bigg\}\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\Bigg]F_2+0+0\,.\end{aligned} \tag{30}$$

International Journal of Advancements in Research & Technology, Volume 1, Issue 3, August-2012
ISSN 2278-7763

$$S^{S}_{ATV}(z,\mu)=\frac{\omega^{AT}(z)}{4}\left[-\frac{1}{2}(r_L r_L^{*}-r_R r_R^{*})\sum_{J=S}^{M}\in_J P_J^S(\mu)T_J^S(\mu_0)\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\right]F_1+$$

$$\frac{\omega^{AT}(z)}{4}\left[-\sum_{J=S}^{M}(-1)^{J+S}\in_J P_J^S(\mu)T_J^S(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]-\frac{1}{2}(r_L r_L^{*}+r_R r_R^{*})\sum_{J=S}^{M}\in_J P_J^S(\mu)T_J^S(\mu_0)\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\right]F_2+0+0. \quad (31)$$

$$S^{S}_{OCL}(z,\mu)=\frac{\omega^{OC}(z)}{4}\left[\sum_{J=S}^{M}(-1)^{J+S}\beta'_J P_J^S(\mu)P_J^S(\mu_{0n})\times\frac{1}{2}(t_L t_L^{*}+t_R t_R^{*})+\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J P_J^S(\mu)R_J^S(\mu_{0n})\right.$$

$$\left.\times\frac{1}{2}(t_L t_L^{*}-t_R t_R^{*})\right]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right)F_1+$$

$$\frac{\omega^{OC}(z)}{4}\left[\sum_{J=S}^{M}(-1)^{J+S}\beta'_J P_J^S(\mu)P_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_L^{*}-t_R t_R^{*}\}\}+\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J P_J^S(\mu)R_J^S(\mu_{0n})\right.$$

$$\left.\times\frac{1}{2}\{t_L t_L^{*}+t_R t_R^{*}\}\}\right]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right)F_2. \quad (32)$$

$$S^{S}_{OCQ}(z,\mu)=\left[\sum_{J=S}^{M}\gamma'_J R_J^S(\mu)P_J^S(\mu_{0n})\times\frac{1}{2}(t_L t_L^{*}+t_R t_R^{*})+\sum_{J=S}^{M}(\alpha'_J R_J^S(\mu)R_J^S(\mu_{0n})+\xi'_J T_J^S(\mu)T_J^S(\mu_{0n}))\right.$$

$$\left.\times\frac{1}{2}(t_L t_L^{*}-t_R t_R^{*})\right].\exp\left(-\frac{(z_\omega)}{\mu_0}\right)\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right).\left(\frac{\mu_0}{\mu_{0n}}\right).F_1+$$

$$\frac{\omega^{OC}(z)}{4}\left[\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J R_J^S(\mu)P_J^S(\mu_{0n})\frac{1}{2}\{t_L t_L^{*}-t_R t_R^{*}\}\}+\right.$$

$$\left.\sum_{J=S}^{M}(-1)^{J+S}(\alpha_J R_J^S(\mu)R_J^S(\mu_{0n})+\xi_J T_J^S(\mu)T_J^S(\mu_{0n}))\frac{1}{2}(t_L t_L^{*}+t_R t_R^{*})\right]\exp\left(-\frac{z_\omega}{\mu_0}\right)\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right)\left(\frac{\mu_0}{\mu_{0n}}\right)F_2. \quad (33)$$

$$S^{S}_{OCU}(z,\mu)=\frac{\omega^{OC}(z)}{4}\left[-\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J T_J^S(\mu)P_J^S(\mu_{0n})\frac{1}{2}\{t_L t_L^{*}+t_R t_R^{*}\}\}-\right.$$

$$\left.\sum_{J=S}^{M}(-1)^{J+S}(\alpha'_J T_J^S(\mu)R_J^S(\mu_{0n})+\xi'_J R_J^S(\mu)T_J^S(\mu_{0n}))\frac{1}{2}\{t_L t_L^{*}-t_R t_R^{*}\}\}\right]$$

$$\exp\left(-\frac{z_\omega}{\mu_0}\right)\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right)\left(\frac{\mu_0}{\mu_{0n}}\right)F_1+\frac{\omega^{OC}(z)}{4}\left[-\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J T_J^S(\mu)P_J^S(\mu_{0n})\frac{1}{2}\{t_L t_L{}^*-t_R t_R{}^*\}\}-\right.$$

$$\left.\sum_{J=S}^{M}(-1)^{J+S}(\alpha'_J T_J^S(\mu)R_J^S(\mu_{0n})+\xi'_J R_J^S(\mu)T_J^S(\mu_{0n}))\frac{1}{2}\{t_L t_L{}^*-t_R t_R{}^*\}\}\right]\exp\left(-\frac{z_\omega}{\mu_0}\right)\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right)\left(\frac{\mu_0}{\mu_{0n}}\right)F_2 \qquad (34)$$

$$S_{OCV}^S(z,\mu)=\frac{\omega^{OC}(z)}{4}\left[-\sum_{J=S}^{M}(-1)^{J+S}\in'_J P_J^S(\mu)T_J^S(\mu_{0n})\frac{1}{2}\{t_L t_L{}^*-t_R t_R{}^*\}\right]\exp\left(-\frac{z_\omega}{\mu_0}\right)\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right)\left(\frac{\mu_0}{\mu_{0n}}\right)F_1$$

$$+\frac{\omega^{OC}(z)}{4}\left[-\sum_{J=S}^{M}(-1)^{J+S}\in'_J P_J^S(\mu)T_J^S(\mu_{0n})\frac{1}{2}\{t_L t_L{}^*+t_R t_R{}^*\}\right]\exp\left(-\frac{z_\omega}{\mu_0}\right)\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right)\left(\frac{\mu_0}{\mu_{0n}}\right)F_2. \qquad (35)$$

The azimuth free source for atmosphere and ocean can now be expressed as

$$S_{AT}^S(z,\mu)=\begin{bmatrix}S_{ATL}^S(z,\mu)\\ S_{ATL}^S(z,\mu)\\ S_{ATL}^S(z,\mu)\\ S_{ATL}^S(z,\mu)\end{bmatrix} \qquad (36) \qquad \text{and} \qquad S_{OC}^S(z,\mu)=\begin{bmatrix}S_{OCL}^S(z,\mu)\\ S_{OCQ}^S(z,\mu)\\ S_{OCU}^S(z,\mu)\\ S_{OCV}^S(z,\mu)\end{bmatrix} \qquad (37)$$

2.2 Discretization of equation of transfer

We shall solve the equation of transfer employing a new version of Chandrasekhar's DOM. First we rewrite (Dropping "s" in intensity) the homogeneous version of the reduced radiative transfer equations for atmosphere and ocean medium from equations (22) and (25).

$$\mu\frac{d}{dz}I_{AT}(z,\mu)+I_{AT}(z,\mu)=\frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}PAT_J^S(z,\mu)\int_{-1}^{+1}\mathbf{P}_J^S(\mu')I_{AT}(z,\mu') \qquad (38)$$

$$\mu\frac{d}{dz}I_{OC}(z,\mu)+I_{OC}(z,\mu)=\frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}POC_J^S(z,\mu)\int_{-1}^{+1}\mathbf{P}_J^S(\mu')I_{OC}(z,\mu') \qquad (39)$$

We shall use the set of separated but coupled non-linear integro differential equations for each component of stokes vector for discretization. We shall use Chandrasekhar discretization [6] scheme to break the continuous radiation field into 2N quadrature directions with corresponding weights keeping optical depth dependence exact, for $\mathbf{i,k=\pm1,\pm2,\pm3,......,\pm N}$. This enable us to form a set of n equations for each $\mathbf{i}$ (negative as well as positive). Each

homogeneous version of equations is then replaced by N equivalent equations, separated for positive and negative quadrature directions in the following form, with corresponding Gaussian
weight functions expressed and interchanging the Gaussian summation with Fourier summation, the equations are written as under.

$$\mathbf{w_k} = \frac{1}{\mathbf{P'^S(\mu_k)}} \int_{-1}^{+1} \frac{\mathbf{P(\mu_k)}}{\mu - \mu_k} \mathbf{d\mu}. \tag{40}$$

$$\mu_i \frac{d}{dz} I_{AT}(z,\mu_i) + I_{AT}(z,\mu_i) = \frac{\omega_{AT}}{2} \sum_{J=S}^{M} PAT_J^S(z,\mu_i) \sum_{k=1}^{N} \omega_k (\mathbf{P}_J^S(\mu_k I_{AT}(z,\mu_K) + \mathbf{P}_k^S(-\mu_k) I_{AT}(z,-\mu_k)) \tag{41}$$

$$-\mu_i \frac{d}{dz} I_{AT}(z,-\mu_i) + I_{AT}(z,-\mu_i) = \frac{\omega_{AT}}{2} \sum_{J=S}^{M} PAT_J^S(z,-\mu_i) \sum_{k=1}^{N} \omega_k (\mathbf{P}_J^S(\mu_k) I_{AT}(z,\mu_K) + \mathbf{P}_k^S(-\mu_k) I_{AT}(z,-\mu_k)) \tag{42}$$

$$\mu_i \frac{d}{dz} I_{OC}(z,\mu_i) + I_{OC}(z,\mu_i) = \frac{\omega_{AT}}{2} \sum_{J=S}^{M} POC_J^S(z,\mu_i) \sum_{k=1}^{N} \omega_k (\mathbf{P}_J^S(\mu_k) I_{OC}(z,\mu_K) + \mathbf{P}_k^S(-\mu_k) I_{OC}(z,-\mu_k)) \tag{43}$$

$$-\mu_i \frac{d}{dz} I_{OC}(z,-\mu_i) + I_{OC}(z,-\mu_i) = \frac{\omega_{AT}}{2} \sum_{J=S}^{M} POC_J^S(z,-\mu_i) \sum_{k=1}^{N} \omega_k (\mathbf{P}_J^S(\mu_k) I_{OC}(z,\mu_K) + \mathbf{P}_k^S(-\mu_k) I_{OC}(z,-\mu_k)) \tag{44}$$

The solution of the polarized radiative transfer problem is assumed as follows:

$$\mathbf{I}_{AT/OC}(z,\pm\mu_i) = \mathbf{H}^{AT/OC}(\gamma,\pm\mu_i)\exp(-\frac{z}{\gamma}). \tag{45}$$

Using the mentioned assumed solution in the following set of equations (41-44) we arrive at the following set of equations for atmosphere and ocean.

$$\left(\mathbf{I} - \frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_{+}^{AT}(\gamma) = \frac{\omega^{AT}(z)}{2} \sum_{J=S}^{M} \mathbf{\Pi}(J,S) B_J^{AT} \mathbf{T}_J^S(AT,\gamma) \tag{46}$$

$$\left(\mathbf{I} + \frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_{-}^{AT}(\gamma) = \frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_J^{AT}\mathbf{T}_J^S(AT,\gamma) \quad (47)$$

$$\left(\mathbf{I} - \frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_{+}^{OC}(\gamma) = \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{OC}\mathbf{T}_J^S(OC,\gamma) \quad (48)$$

$$\left(\mathbf{I} + \frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_{-}^{OC}(\gamma) = \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_J^{OC}\mathbf{T}_J^S(OC,\gamma) \quad (49)$$

Where,

$$\mathbf{\Pi}(J,S) = \left[\mathbf{P}_J^S(\mu_1),\mathbf{P}_J^S(\mu_2),\mathbf{P}_J^S(\mu_3),\ldots,\mathbf{P}_J^S(\mu_N)\right]^T \quad (50)$$

$$\mathbf{T}_J^S(AT/OC,\gamma) = \mathbf{\Pi}^T(J,S)\mathbf{\Sigma}\mathbf{H}_{+}^{AT/OC}(\gamma) + (-1)^{J-S}D\mathbf{\Pi}^T(J,S)\mathbf{\Sigma}\mathbf{H}_{-}^{AT/OC}(\gamma). \quad (51)$$

$\mathbf{NA}^{AT} = \mathbf{H}_{+}^{AT}(\gamma) + \mathbf{H}_{-}^{AT}(\gamma)$; (52a) $\mathbf{NB}^{AT} = \mathbf{H}_{+}^{AT}(\gamma) - \mathbf{H}_{-}^{AT}(\gamma)$ (52b)

$\mathbf{NA}^{OC} = \mathbf{H}_{+}^{OC}(\gamma) + \mathbf{H}_{-}^{OC}(\gamma)$; (53a) $\mathbf{NB}^{OC} = \mathbf{H}_{+}^{OC}(\gamma) - \mathbf{H}_{-}^{OC}(\gamma)$ (53b)

$\mathbf{A}^{AT}\mathbf{X}^{AT} = \frac{1}{\gamma}\mathbf{Y}^{AT}$; (54) $\mathbf{B}^{AT}\mathbf{Y}^{AT} = \frac{1}{\gamma}\mathbf{X}^{AT}$. (55)

$\mathbf{A}^{OC}\mathbf{X}^{OC} = \frac{1}{\gamma}\mathbf{Y}^{OC}$; (56) $\mathbf{B}^{OC}\mathbf{Y}^{OC} = \frac{1}{\gamma}\mathbf{X}^{OC}$. (57)

$\mathbf{X}^{AT/OC}$ are the eigen vectors and γ is the eigen value of the above equations. The matrices $\mathbf{A}^{AT/OC}$ and $\mathbf{B}^{AT/OC}$ are evaluated using the given equations

$$\mathbf{A}^{AT} = \left(\mathbf{I} - \frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{AT}[I + (-1)^{J-S}D]\mathbf{\Pi}(J,S)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1}, \quad (58)$$

$$\mathbf{B}^{AT} = \left(\mathbf{I} - \frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{AT}[I - (-1)^{J-S}D]\mathbf{\Pi}(J,S)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1}, \quad (59)$$

$$\mathbf{A}^{OC} = \left(\mathbf{I} - \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{OC}[I + (-1)^{J-S}D]\mathbf{\Pi}(J,S)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1}, \quad (60)$$

$$\mathbf{B}^{OC} = \left(\mathbf{I} - \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{OC}[I - (-1)^{J-S}D]\mathbf{\Pi}(J,S)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1}. \quad (61)$$

$\mathbf{X}^{AT} = \mathbf{X}\left[\mathbf{NA}^{AT}\right]$ (62) $\mathbf{Y}^{AT} = \mathbf{X}\left[\mathbf{NB}^{AT}\right]$. (63)

$\mathbf{X}^{OC} = \mathbf{X}\left[\mathbf{NA}^{OC}\right]$; (64) $\mathbf{Y}^{OC} = \mathbf{X}\left[\mathbf{NB}^{OC}\right]$. (65)

International Journal of Advancements in Research & Technology, Volume 1, Issue 3, August-2012
ISSN 2278-7763

$$(\mathbf{B}^{AT}\mathbf{A}^{AT})\mathbf{X}^{AT} = \Im^{AT}\mathbf{X}^{AT}; \quad (66)$$

$$(\mathbf{B}^{OC}\mathbf{A}^{OC})\mathbf{X}^{OC} = \Im^{OC}\mathbf{X}^{OC} \quad (67)$$

$$(\mathbf{A}^{AT}\mathbf{B}^{AT})\mathbf{Y}^{AT} = \Im_{AT}\mathbf{Y}^{AT}; \quad (68)$$

$$(\mathbf{A}^{OC}\mathbf{B}^{OC})\mathbf{Y}^{OC} = \Im_{OC}\mathbf{Y}^{OC} \quad (69)$$

$$\Im = \frac{1}{\gamma^2}. \quad (70)$$

2.3 Numerical examples:

$$\mu_{\pm 1} = \pm 0.3399810; \ \mu_{\pm 2} = \pm 0.8611363; \quad (71)$$

$$\omega_1 = \omega_{-1} = 0.6521452; \quad \omega_2 = \omega_{-2} = 0.3478548 \quad (72)$$

$$\mathbf{\Pi}(0,0) = \left[\mathrm{diag}\{1,0,0,1\}, \mathrm{diag}\{1,0,0,1\}\right]^{T}; \quad (73)$$

$$\mathbf{\Pi}(1,0) = \left[\mathrm{diag}\{\mu_1,0,0,\mu_1\}, \mathrm{diag}\{\mu_2,0,0,\mu_2\}\right]^{T}; \quad (74)$$

$$\mathbf{\Pi}(2,0) = \left[\mathbf{P}_2^0(\mu_1), \mathbf{P}_2^0(\mu_2)\right]^{T} \quad (75)$$

$$\mathbf{\Pi}(2,0) = \left[\begin{pmatrix} -0.3266 & 0 & 0 & 0 \\ 0 & 0.5416 & 0 & 0 \\ 0 & 0 & 0.5416 & 0 \\ 0 & 0 & 0 & -0.3266 \end{pmatrix}, \begin{pmatrix} 0.6123 & 0 & 0 & 0 \\ 0 & 0.1583 & 0 & 0 \\ 0 & 0 & 0.1583 & 0 \\ 0 & 0 & 0 & 0.6123 \end{pmatrix}\right]^{T} \quad (76)$$

we have used the values of basic constants for spherical particles for ocean water Vestrucci and Siewert [7] considering mie scattering of light, with wavelength 0.951 micrometer having gamma distribution with effective radius 0.2 micro meters, effective variance 0.07 and refractive index 1.34. In table 2 the basic constants for oblate spheroids particle for atmosphere as measured by Wauben and Hovenier [8] have been used.

Table 1:

Basic constants for spherical particle in ocean

l	α_l	β_l	γ_l	δ_l	ε_l	ζ_l
0	0.0	1.0	0.0	0.7120634246	0.0	0.0
1	0.0	1.4552931819	0.0	1.761411931	0.0	0.0
2	3.3091220464	1.0540263128	-0.7552491518	1.0668243107	0.420726875	2.5773207443
3	0.9633758276	0.3975899378	-0.36619934319	0.3965110389	0.0850671555	0.7574437604
4	0.2474124256	0.1165930161	-0.1155748816	0.0957641237	0.0154318420	0.1638177665
5	0.0452636955	0.0238747702	-0.0249815879	0.0176508810	0.0031534874	0.0278314781
6	0.0068892608	0.0039501033	-0.0041675362	0.0026154886	0.0004010299	0.0038897248
7	0.0008798202	0.0005388807	-0.0005739043	0.0003271332	0.0000460147	0.0004642654
8	0.0000987255	0.0000637172	-0.0000677900	0.0000358314	0.0000042875	0.0000490224
9	0.0000099020	0.0000066697	-0.0000070926	0.0000035142	0.0000003617	0.0000046647
10	0.0000009071	0.0000006329	-0.0000006712	0.0000003148	0.0000000272	0.0000004075
11	0.0000000769	0.0000000553	-0.0000000585	0.0000000261	0.0000000019	0.0000000331
12	0.0000000061	0.0000000045	-0.0000000047	0.0000000020	0.0000000001	0.0000000025
13	0.0000000005	0.000000003	-0.0000000004	0.0000000001	0.0000000000	0.0000000002

International Journal of Advancements in Research & Technology, Volume 1, Issue 3, August-2012
ISSN 2278-7763

Table 2:

Basic Constants for oblate spheroid particle in atmosphere

l	α_l	β_l	γ_l	δ_l	ε_l	ζ_l
0	0.0	1.0	0.0	0.915207	0.0	0.0
1	0.0	2.104031	0.0	2.095727	0.0	0.0
2	3.726079	2.095158	-0.116688	2.008624	0.065456	3.615946
3	2.202868	1.414939	-0.209370	1.436545	0.221658	2.240516
4	1.190694	0.703593	-0.227137	0.706244	0.097752	1.139473
5	0.391203	0.235001	-0.144524	0.238475	0.052458	0.365605
6	0.105556	0.064039	-0.052640	0.056448	0.009239	0.082779
7	0.020484	0.012837	-0.012400	0.009703	0.001411	0.013649
8	0.003097	0.000246	-0.002093	0.001267	0.000133	0.001721
9	0.000366	0.000246	-0.000267	0.000130	0.00001	0.000172
10	0.000035	0.000024	-0.000027	0.000011	0.000001	0.000014
11	0.000003	0.000002	-0.000002	0.000001	0.000000	0.000001

We further consider,

$$\mathbf{H}_{AT/OC}(\pm)= \left[I_{AT/OC}(z,\pm\mu_1)^T, I_{AT/OC}(z,\pm\mu_2)^T, I_{AT/OC}(z,\pm\mu_3)^T, \ldots\ldots, I_{AT/OC}(z,\pm\mu_N)^T\right]^T \quad (77)$$

$$\mathbf{H}_{AT}(+)=\sum_{j=1}^{4N} A_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\exp\left(-\frac{z}{\gamma_j^{AT}}\right)+B_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_w-z}{\gamma_j^{AT}}\right) \quad (78)$$

$$\mathbf{H}_{OC}(+)=\sum_{j=1}^{4N} A_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right)+B_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1-z}{\gamma_j^{OC}}\right) \quad (79)$$

$$\mathbf{H}_{AT}(-)=\mathbf{\Delta}\sum_{j=1}^{4N} A_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\exp\left(-\frac{z}{\gamma_j^{AT}}\right)+B_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega-z}{\gamma_j^{AT}}\right) \quad (80)$$

$$\mathbf{H}_{OC}(-)=\boldsymbol{\Delta}\sum_{j=1}^{4N} A_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right)+B_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1-z}{\gamma_j^{OC}}\right) \tag{81}$$

Since both the eigenvector and eigenvalues may be complex we want to write equations (78-81) in terms of real quantities. Let us represent Nr for number of real separation constants and Nc for the number of complex pairs of separation constants. With these we rewrite equations (78-81) as

$$RE_+^{AT}(z)=\sum_{j=1}^{Nr} A_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\exp\left(-\frac{z}{\gamma_j^{AT}}\right)+B_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega-z}{\gamma_j^{AT}}\right), \tag{82}$$

$$RE_+^{OC}(z)=\sum_{j=1}^{Nr} A_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right)+B_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1-z}{\gamma_j^{AT/OC}}\right), \tag{83}$$

$$CO_+^{AT}(z)=\sum_{j=1}^{Nc}\sum_{\alpha=1}^{2} A_j^{AT(\alpha)}F_+^{AT(\alpha)}(z,\gamma_j^{AT})+B_j^{AT(\alpha)}F_-^{AT(\alpha)}(z_\omega-z,\gamma_j^{AT}), \tag{84}$$

$$CO_+^{OC}(z)=\sum_{j=1}^{Nc}\sum_{\alpha=1}^{2} A_j^{OC(\alpha)}F_+^{OC(\alpha)}(z,\gamma_j^{OC})+B_j^{OC(\alpha)}F_-^{OC(\alpha)}(z_1-z,\gamma_j^{OC}), \tag{85}$$

$$RE_-^{AT}(z)=\boldsymbol{\Delta}\sum_{j=1}^{Nr} A_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\exp\left(-\frac{z}{\gamma_j^{AT}}\right)+B_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega-z}{\gamma_j^{AT}}\right), \tag{86}$$

$$RE_-^{OC}(z)=\boldsymbol{\Delta}\sum_{j=1}^{Nr} A_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right)+B_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1-z}{\gamma_j^{OC}}\right), \tag{87}$$

$$CO_-^{AT}(z)=\sum_{j=1}^{Nc}\sum_{\alpha=1}^{2} A_j^{AT(\alpha)}F_-^{AT(\alpha)}(z,\gamma_j^{AT})+B_j^{AT(\alpha)}F_+^{AT(\alpha)}(z_\omega-z,\gamma_j^{AT}), \tag{88}$$

$$CO_-^{OC}(z)=\sum_{j=1}^{Nc}\sum_{\alpha=1}^{2} A_j^{OC(\alpha)}F_-^{OC(\alpha)}(z,\gamma_j^{OC})+B_j^{OC(\alpha)}F_+^{OC(\alpha)}(z_1-z,\gamma_j^{OC}), \tag{89}$$

Where,

$$F_\pm^{AT/OC(1)}(z,\gamma_j^{AT/OC})=\mathrm{Re}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_\pm^{AT/OC}(\gamma_j^{AT/OC})\right\}-\mathrm{Im}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_\pm^{AT/OC}(\gamma_j^{AT/OC})\right\}$$

$$= F^{AT/OC}{}_{1(\pm1)} - F^{AT/OC}{}_{1(\pm2)}. \tag{90}$$

$$F_{\pm}^{AT/OC(2)}(z,\gamma_j^{AT/OC}) = \mathrm{Im}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{AT/OC}(\gamma_j^{AT/OC})\right\} - \mathrm{Re}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{AT/OC}(\gamma_j^{AT/OC})\right\}.$$

$$= F^{AT/OC}{}_{2(\pm1)} - F^{AT/OC}{}_{2(\pm2)}. \tag{91}$$

We shall use following notations for different values of z.

$$F_{\pm}^{AT(1)}(z_\omega,\gamma_j^{AT}) = \mathrm{Re}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\ \right\} - \mathrm{Im}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\} = F^{AT}{}_{11(\pm1)} - F^{AT}{}_{11(\pm2)}. \tag{92}$$

$$F_{\pm}^{AT(1)}(0,\gamma_j^{AT}) = \mathrm{Re}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\ \right\} - \mathrm{Im}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\}$$

$$= \mathrm{Re}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\ \right\} - \mathrm{Im}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\} = F^{AT}{}_{12(\pm1)} - F^{AT}{}_{12(\pm2)}. \tag{93}$$

$$F_{\pm}^{AT(2)}(z_\omega,\gamma_j^{AT}) = \mathrm{Im}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\} - \mathrm{Re}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\} = F^{AT}{}_{21(\pm1)} - F^{AT}{}_{21(\pm2)}. \tag{94}$$

$$F_{\pm}^{AT(2)}(0,\gamma_j^{AT}) = \mathrm{Re}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\} - \mathrm{Im}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\} = F^{AT}{}_{22(\pm1)} - F^{AT}{}_{22(\pm2)} \tag{95}$$

$$F_{\pm}^{OC(1)}(z_\omega,\gamma_j^{OC}) = \mathrm{Re}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\right\} - \mathrm{Im}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC}\right\} = F^{OC}{}_{11(\pm1)} - F^{OC}{}_{11(\pm2)}. \tag{96}$$

$$F_{\pm}^{OC(1)}(z_1 - z_\omega,\gamma_j^{OC}) = \mathrm{Re}\left\{\exp\left(-\frac{z_1 - z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\right\} - \mathrm{Im}\left\{\exp\left(-\frac{z_1 - z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC}\right\} = F^{OC}{}_{12(\pm1)} - F^{OC}{}_{12(\pm2)}. \tag{97}$$

$$F_{\pm}^{OC(2)}(z_\omega,\gamma_j^{OC}) = \mathrm{Im}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\right\} - \mathrm{Re}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC}\right\} = F^{OC}{}_{21(\pm1)} - F^{OC}{}_{21(\pm2)}. \tag{98}$$

$$F_{\pm}^{OC(2)}(z_1 - z_\omega, \gamma_j^{OC}) = \mathrm{Im}\left\{\exp\left(-\frac{z_1 - z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\right\} - \mathrm{Re}\left\{\exp\left(-\frac{z_1 - z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\right\} = F^{OC}22(\pm1) - F^{OC}22(\pm2). \tag{100}$$

$$F_{\pm}^{OC(1)}(z_1, \gamma_j^{OC}) = \mathrm{Re}\left\{\exp\left(-\frac{z_1}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\right\} - \mathrm{Im}\left\{\exp\left(-\frac{z_1}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\right\} = \left[\, F^{OC}13(\pm1) - F^{OC}13(\pm2) \,\right] \tag{101}$$

$$F_{\pm}^{OC(2)}(z_1, \gamma_j^{OC}) = \mathrm{Im}\left\{\exp\left(-\frac{z_1}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\right\} - \mathrm{Re}\left\{\exp\left(-\frac{z_1}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\right\} = \left[\, F^{OC}23(\pm1) - F^{OC}23(\pm2) \,\right] \tag{102}$$

3. PARTICULAR & GENERAL SOLUTION

3.1 The Infinite Medium Green's Function:

In order to develop particular solution we shall proceed with the approach developed in (Barichello, L. B., Garcia, RDM, Siewert, C.E.) [9] to accommodate for the inhomogeneous source term $S^{J}_{AT/OC}(z,\mu)$ that appears in (36 &37). The elementary solutions

developed in the previous chapter can now be used to construct the green functions which in turn are required to find the particular solution i.e. to express it in terms of infinite-medium Green's function and therefore we take up the following two problems as mentioned.

For atmosphere in regard to Green function $\mathbf{M}^{AT}(z,\pm\mu_i : x,\mu_k)$ & $\mathbf{M}^{AT}(z,\pm\mu_i : x,-\mu_k)$ for any source location x within atmosphere, we have

$$\left(\mu_i \frac{d}{dz} + I\right)\mathbf{M}^{AT/OC}(z,\mu_i : x,\mu_k) = \frac{\omega^{AT/OC}(z)}{2}\sum_{J=S}^{M}\mathbf{K}^{j}_{AT/OC}(z,\mu_i : x,\mu_k) + I\delta(z-x)\delta_{i,k} \tag{103}$$

$$\left(-\mu_i \frac{d}{dz} + I\right)\mathbf{M}^{AT/OC}(z,-\mu_i : x,\mu_k) = \frac{\omega^{AT/OC}(z)}{2}\sum_{J=S}^{M}\mathbf{K}^{j}_{AT/OC}(z,-\mu_i : x,\mu_k) \tag{104}$$

$$\left(\mu_i \frac{d}{dz} + I\right)\mathbf{M}^{AT/OC}(z,\mu_i : x,-\mu_k) = \frac{\omega^{AT/OC}(z)}{2}\sum_{J=S}^{M}\mathbf{K}^{j}_{AT/OC}(z,\mu_i : x,-\mu_k) \tag{105}$$

$$\left(-\mu_i \frac{d}{dz}+I\right)\mathbf{M}^{AT/OC}(z,-\mu_i : x,-\mu_k) = \frac{\omega^{AT/OC}(z)}{2}\sum_{J=S}^{M}\mathbf{K}^{j}_{AT/OC}(z,-\mu_i : x,-\mu_k) + I\delta(z-x)\delta_{i,k} \tag{106}$$

$$\mathbf{K}^{J}_{AT/OC}(z,\pm\mu_i : x,\pm\mu_k) = \mathbf{P}^{S}_{J}(\pm\mu_i)B^{AT/OC}_{J}\sum_{\beta=1}^{N} w_\beta \mathbf{M}^{AT/OC}_{S,\beta}(z,x,\pm\mu_k) \tag{107}$$

$$\mathbf{M}^{AT/OC}_{S,\beta}(z : x,\pm\mu_k) = \mathbf{P}^{S}_{J}(\mu_\beta)\mathbf{M}^{AT/OC}(z,\mu_\beta : x,\pm\mu_k) + \mathbf{P}^{S}_{J}(-\mu_\beta)\mathbf{M}^{AT/OC}(z,-\mu_\beta : x,\pm\mu_k) \tag{108}$$

Source is located at $x \in (0, z_w)$ for atmosphere and at $x \in (z_w, z_1)$ in the ocean along with the source direction defined by $\mu_k \in \{\mu_i\}$. We have defined $\delta(z-x)$ as the Dirac delta "function" and $\delta_{i,k}$ as the Kronecker delta. Inclusion of the identity matrix in source term of equations (103) and (106) clearly indicates that each of the two Green's functions must come out as a (4X4) matrix.

We now proceed to develop the solution for $\mathbf{M}^{AT}(z,\xi_\beta : x,\pm\mu_k)$ by following Case and Zweifel [10].We can write one solution bounded as $z \to \infty$ and valid in the region $z > x$ for the homogeneous equation and another solution valid for $z < x$ and bounded as $z \to -\infty$). For matching up these two solutions with the notion of "jump" condition, valid when $i,k = 1,2,...,N$, we write, for atmosphere and ocean respectively

1. (+, +) equations $$\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{AT/OC}(z+\varepsilon,\mu_i : x,\mu_k) - \mathbf{M}^{AT/OC}(z-\varepsilon,\mu_i : x,\mu_k)] = I\delta_{i,k} \tag{109}$$

2. (–, +) equations $$-\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{AT/OC}(z+\varepsilon,-\mu_i : x,\mu_k) - \mathbf{M}^{AT/OC}(z-\varepsilon,-\mu_i : x,\mu_k)] = 0 \tag{110}$$

3. (+,–) equations $$\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{AT/OC}(z+\varepsilon,\mu_i : x,-\mu_k) - \mathbf{M}^{AT/OC}(z-\varepsilon,\mu_i : x,-\mu_k)] = 0 \tag{111}$$

4. (–,–) equations $$-\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{AT/OC}(z+\varepsilon,-\mu_i : x,-\mu_k) - \mathbf{M}^{AT/OC}(z-\varepsilon,-\mu_i : x,-\mu_k)] = I\delta_{i,k} \tag{112}$$

Case A: For $z > x$:

$$\mathbf{M}^{AT/OC}_{+}(z : x,\pm\mu_k) = \sum_{J=1}^{4N}\mathbf{H}^{AT/OC}_{+}(\gamma^{AT/OC}_{J})C^{AT/OC}{}_{j}(\pm\mu_k)\exp\left(-\frac{(z-x)}{\gamma^{AT/OC}_{J}}\right), \quad z > x, \tag{113}$$

$$\mathbf{M}^{AT/OC}_{-}(z : x,\pm\mu_k) = \mathbf{\Delta}\sum_{J=1}^{4N}\mathbf{H}^{AT/OC}_{-}(\gamma^{AT/OC}_{J})C^{AT/OC}{}_{j}(\pm\mu_k)\exp\left(-\frac{(z-x)}{\gamma^{AT/OC}_{J}}\right), \quad z > x, \tag{114}$$

Case B: For z<x:

$$\mathbf{M}^{AT/OC}_{+}(z : x,\pm\mu_k) = -\sum_{J=1}^{4N}\mathbf{H}^{AT/OC}_{-}(\gamma^{AT/OC}_{J})D^{AT/OC}{}_{j}(\pm\mu_k)\exp\left(-\frac{(x-z)}{\gamma^{AT/OC}_{J}}\right), \quad z < x, \tag{115}$$

$$\mathbf{M}_{-}^{AT/OC}(z:x,\pm\mu_k) = -\Delta\sum_{J=1}^{4N}\mathbf{H}_{+}^{AT/OC}(\gamma_J^{AT/OC})D^{AT/OC}{}_{j}(\pm\mu_k)\exp\left(-\frac{(x-z)}{\gamma_J^{AT/OC}}\right), \quad z<x, \tag{116}$$

$$C^{AT/OC}{}_{j}(\pm\mu_k) = [C^{AT/OC}{}_{1,j}(\pm\mu_k), C^{AT/OC}{}_{2,j}(\pm\mu_k), C^{AT/OC}{}_{3,j}(\pm\mu_k), C^{AT/OC}{}_{4,j}(\pm\mu_k)], \tag{117}$$

$$D^{AT/OC}{}_{j}(\pm\mu_k) = [D^{AT/OC}{}_{1,j}(\pm\mu_k), D^{AT/OC}{}_{2,j}(\pm\mu_k), D^{AT/OC}{}_{3,j}(\pm\mu_k), D^{AT/OC}{}_{4,j}(\pm\mu_k)], \tag{118}$$

$$\mathbf{AH}_{+}^{AT}(\gamma_k)^T W\mathbf{XH}_{+}^{AT}(\gamma_j) - \mathbf{AH}_{-}^{AT}(\gamma_k)^T W\mathbf{XH}_{-}^{AT}(\gamma_j) = 0, \quad \gamma_j \neq \gamma_k. \tag{119}$$

$$[\mathbf{AH}_{-}^{AT}(\gamma_k)]^T W\mathbf{XH}_{-}^{AT}(\gamma_j) - [\mathbf{AH}_{-}^{AT}(\gamma_k)]^T W\mathbf{XH}_{+}^{AT}(\gamma_j) = 0. \tag{120}$$

$$C_j^{AT}(\mu_\alpha) = \frac{1}{NAT(\gamma_j)}[\mathbf{AH}_{+}^{AT}(\gamma_j)]^T WR_\alpha \tag{121}$$

$$NAT(\gamma_j) = [\mathbf{AH}_{+}^{AT}(\gamma_j)]^T W\mathbf{XH}_{+}^{AT}(\gamma_j) - [\mathbf{AH}_{-}^{AT}(\gamma_j)]^T W\mathbf{XH}_{-}^{AT}(\gamma_j). \tag{122}$$

$$\mathbf{AH}_{+}^{OC}(\gamma_k)^T W\mathbf{XH}_{+}^{OC}(\gamma_j) - \mathbf{AH}_{-}^{OC}(\gamma_k)^T W\mathbf{XH}_{-}^{OC}(\gamma_j) = 0, \tag{123}$$

$$[\mathbf{AH}_{-}^{OC}(\nu_k)]^T W\mathbf{XH}_{-}^{OC}(\nu_j) - [\mathbf{AH}_{-}^{OC}(\nu_k)]^T W\mathbf{XH}_{+}^{OC}(\nu_j) = 0. \tag{124}$$

$$C_j^{OC}(\mu_\alpha) = \frac{1}{NOC(\gamma_j)}[\mathbf{AH}_{+}^{OC}(\gamma_j)]^T WR_\alpha \tag{125}$$

$$NOC(\gamma_j) = [\mathbf{AH}_{+}^{OC}(\gamma_j)]^T W\mathbf{XH}_{+}^{OC}(\gamma_j) - [\mathbf{AH}_{-}^{OC}(\gamma_j)]^T W\mathbf{XH}_{-}^{OC}(\gamma_j). \tag{126}$$

With the help of green functions developed earlier for z>x and z<x, we can immediately write one particular solution in the following manner

$$I_{AT}^{p}(+;z) = \sum_{\alpha=1}^{N}\int_0^{z_\omega}[\mathbf{M}_{+}^{AT}(z:x,\mu_\alpha)S_{AT}(x,\mu_\alpha) + \mathbf{M}_{+}^{AT}(z:x,-\mu_\alpha)S_{AT}(x,-\mu_\alpha)]\,dx \tag{127}$$

$$I^{p}_{AT}(-;z)=\sum_{\alpha=1}^{N}\int_{0}^{z_{\omega}}[\mathbf{M}_{-}^{AT}(z:x,\mu_{\alpha})S_{AT}(x,\mu_{\alpha})+\mathbf{M}_{-}^{AT}(z:x,-\mu_{\alpha})S_{AT}(x,-\mu_{\alpha})]\,dx \tag{128}$$

Substituting from equations (113) and (115), we can rewrite equation (127 & 128) as

$$I^{p}_{AT/OC}(+;z)=\sum_{j=1}^{4N}\left[\Re_{j}^{AT/OC}(z)\mathbf{H}_{+}^{AT/OC}(\gamma_{j})+\aleph_{j}^{AT/OC}(z)\mathbf{H}_{-}^{AT/OC}(\gamma_{j})\right] \tag{129}$$

$$I^{p}_{AT/OC}(-;z)=\mathbf{\Delta}\sum_{j=1}^{4N}[\Re_{j}^{AT/OC}(z)\mathbf{H}_{-}^{AT/OC}(\gamma_{j})+\aleph_{j}^{AT/OC}(z)\mathbf{H}_{+}^{AT/OC}(\gamma_{j})] \tag{130}$$

Where,

$$\Re_{j}^{AT}(z)=\int_{0}^{z}a_{j}^{AT}(x)\exp\left(-\frac{z-x}{\gamma_{j}}\right)dx \quad \text{and} \quad \aleph_{j}^{AT}(z)=\int_{z}^{z_{\omega}}b_{j}^{AT}(x)\exp\left(-\frac{x-z}{\gamma_{j}}\right)dx. \tag{131}$$

$$a_{j}^{AT}(x)=\frac{1}{NAT(\gamma_{j})}\left[[\mathbf{AH}_{+}^{AT}(\gamma_{j})]^{T}WS_{AT(+)}(x)+[\mathbf{AH}_{-}^{AT}(\gamma_{j})]^{T}W\mathbf{\Delta}S_{AT(-)}(x)\right] \tag{132}$$

$$b_{j}^{AT}(x)=\frac{1}{NAT(-\gamma_{j})}\left[[\mathbf{AH}_{-}^{AT}(\gamma_{j})]^{T}WS_{AT(+)}(x)+[\mathbf{AH}_{+}^{AT}(\gamma_{j})]^{T}W\mathbf{\Delta}S_{AT(-)}(x)\right] \tag{133}$$

$$\Re_{j}^{OC}(z)=\int_{0}^{z}a_{j}^{OC}(x)\exp\left(-\frac{z-x}{\gamma_{j}}\right)dx \quad \text{and} \quad \aleph_{j}^{OC}(z)=\int_{z}^{z_{1}}b_{j}^{OC}(x)\exp\left(-\frac{x-z}{\gamma_{j}}\right)dx. \tag{134}$$

$$a_{j}^{OC}(x)=\frac{1}{NOC(\gamma_{j})}\left[[\mathbf{AH}_{+}^{OC}(\gamma_{j})]^{T}WS_{OC(+)}(x)+[\mathbf{AH}_{-}^{OC}(\gamma_{j})]^{T}W\mathbf{\Delta}S_{OC(-)}(x)\right] \tag{135}$$

$$b_{j}^{OC}(x)=\frac{1}{NOC(-\gamma_{j})}\left[[\mathbf{AH}_{-}^{OC}(\gamma_{j})]^{T}WS_{OC\,(+)}(x)+[\mathbf{AH}_{+}^{OC}(\gamma_{j})]^{T}W\mathbf{\Delta}S_{OC\,(-)}(x)\right]; \tag{136}$$

$$S_{AT/OC\,(\pm)}(x)=\left[[S_{AT/OC}(x,\pm\mu_{1})]^{T},[S_{AT/OC}(x,\pm\mu_{2})]^{T},...,[S_{AT/OC}(x,\pm\mu_{N})]^{T}\right]^{T}; \tag{137}$$

$$S_{AT(\pm)}(x)=S_{AT}(\pm)\exp\left(-\frac{x}{\mu_{0}}\right) \tag{138}$$

$$S_{OC\,(\pm)}(x)=S_{OC}(\pm)\exp\left(-\frac{x}{\mu_{0n}}\right) \tag{139}$$

We shall now consider the case of complex separation constants. For real quantities if we let quantities with asterisks as complex conjugates

$$AZ_{\pm}^{AT/OC}(z,\gamma_{j}^{AT/OC})=\Re_{j}^{AT/OC}(z)\mathbf{H}_{\pm}^{AT/OC}(\gamma_{j}^{AT/OC})+\Re_{j}^{AT/OC\,*}(z)\mathbf{H}_{\pm}^{AT/OC\,*}(\gamma_{j}^{AT/OC}); \tag{140}$$

$$BZ_{\pm}^{AT/OC}(z,\lambda_{j}^{AT/OC})=\aleph_{j}^{AT/OC}(z)\mathbf{H}_{\pm}^{AT/OC}(\gamma_{j}^{AT/OC})+\aleph_{j}^{AT/OC\,*}(z)\mathbf{H}_{\pm}^{AT/OC\,*}(\gamma_{j}^{AT/OC}). \tag{141}$$

International Journal of Advancements in Research & Technology, Volume 1, Issue 3, August-2012
ISSN 2278-7763

The complete particular solution can now be written as

$$I_{AT}^{p}(+;z) = \sum_{j=1}^{Nr}[\Re_j^{AT}(z)\mathbf{H}_+^{AT}(\gamma_j) + \aleph_j^{AT}(z)\mathbf{H}_-^{AT}(\gamma_j)] + \sum_{j=1}^{Nc} AZ_+^{AT}(z,\gamma_j^{AT}) + BZ_-^{AT}(z,\gamma_j^{AT}) \tag{142}$$

$$I_{AT}^{p}(-;z) = \mathbf{\Delta}\sum_{j=1}^{Nr}[\Re_j^{AT}(z)\mathbf{H}_-^{AT}(\gamma_j) + \aleph_j^{AT}(z)\mathbf{H}_+^{AT}(\gamma_j)] + \mathbf{\Delta}\sum_{j=1}^{Nc} AZ_-^{AT}(z,\gamma_j^{AT}) + BZ_+^{AT}(z,\lambda_j^{AT}) \tag{143}$$

$$I_{OC}^{p}(+;z) = \sum_{j=1}^{Nr}[\Re_j^{OC}(z)\mathbf{H}_+^{OC}(\gamma_j) + \aleph_j^{OC}(z)\mathbf{H}_-^{OC}(\gamma_j)] + \sum_{j=1}^{Nc} AZ_+^{OC}(z,\gamma_j^{OC}) + BZ_-^{OC}(z,\lambda_j^{OC}) \tag{144}$$

$$I_{OC}^{p}(-;z) = \mathbf{\Delta}\sum_{j=1}^{Nr}[\Re_j^{OC}(z)\mathbf{H}_-^{OC}(\gamma_j) + \aleph_j^{OC}(z)\mathbf{H}_+^{OC}(\gamma_j)] + \mathbf{\Delta}\sum_{j=1}^{Nc} [AZ_-^{OC}(z,\gamma_j^{OC}) + BZ_+^{OC}(z,\lambda_j^{OC})] \tag{145}$$

We now find out appropriate expressions of the particular solutions suitable for application in the boundary conditions.

For top boundary condition the following particular form will be used by setting z=0 in (142)

$$I_{AT}^{p}(-;0) = \mathbf{\Delta}\sum_{j=1}^{Nr}[\Re_j^{AT}(0)\mathbf{H}_-^{AT}(\gamma_j) + \aleph_j^{AT}(0)\mathbf{H}_+^{AT}(\gamma_j)] + \sum_{j=1}^{Nc} AZ_+^{AT}(0,\gamma_j^{AT}) + BZ_-^{AT}(0,\gamma_j^{AT}) \tag{146}$$

The particular form of solutions for application in the second and third boundary conditions are given by

$$[I_{AT}^{p}(+;z_\omega)] = \sum_{j=1}^{Nr}[\Re_j^{AT}(z_\omega)\mathbf{H}_+^{AT}(\gamma_j) + \aleph_j^{AT}(z_\omega)\mathbf{H}_-^{AT}(\gamma_j)] + \sum_{j=1}^{Nc} AZ_+^{AT}(z_\omega,\gamma_j^{AT}) + BZ_-^{AT}(z_\omega,\gamma_j^{AT}) \tag{147}$$

$$[I_{AT}^{p}(-;z_\omega)] = \mathbf{\Delta}\sum_{j=1}^{Nr}[\Re_j^{AT}(z_\omega)\mathbf{H}_-^{AT}(\gamma_j) + \aleph_j^{AT}(z_\omega)\mathbf{H}_+^{AT}(\gamma_j)] + \mathbf{\Delta}\sum_{j=1}^{Nc} AZ_-^{AT}(z_\omega,\gamma_j^{AT}) + BZ_+^{AT}(z_\omega,\lambda_j^{AT}) \tag{148}$$

$$[I_{OC}^{p}(+;z_\omega)] = \sum_{j=1}^{Nr}[\Re_j^{OC}(z_\omega)\mathbf{H}_+^{OC}(\gamma_j) + \aleph_j^{OC}(z_\omega)\mathbf{H}_-^{OC}(\gamma_j)] + \sum_{j=1}^{Nc} AZ_+^{OC}(z_\omega,\gamma_j^{OC}) + BZ_-^{OC}(z_\omega,\lambda_j^{OC}) \tag{149}$$

$$[I_{OC}^{p}(-;z_\omega)] = \mathbf{\Delta}[\sum_{j=1}^{Nr}\Re_j^{OC}(z_\omega)\mathbf{H}_-^{OC}(\gamma_j) + \aleph_j^{OC}(z_\omega)\mathbf{H}_+^{OC}(\gamma_j)] + \mathbf{\Delta}\sum_{j=1}^{Nc}[AZ_-^{OC}(z_\omega,\gamma_j^{OC}) + BZ_+^{OC}(z\omega,\lambda_j^{OC})] \tag{150}$$

The particular solution that will be used in the bottom boundary condition

International Journal of Advancements in Research & Technology, Volume 1, Issue 3, August-2012
ISSN 2278-7763

$$[I^{p}_{OC}(+;z_1)] = \sum_{j=1}^{Nr} \Re^{OC}_{j}(z_1)\mathbf{H}^{OC}_{+}(\gamma_j) + \aleph^{OC}_{j}(z_1)\mathbf{H}^{OC}_{-}(\gamma_j) + \sum_{j=1}^{Nc} AZ^{OC}_{+}(z_1,\gamma^{OC}_{j}) + BZ^{OC}_{-}(z_1,\lambda^{OC}_{j}) \tag{151}$$

We also have

$$\Re^{AT}_{j}(0) = \mu_0\gamma^{AT}_{j}a^{AT}_{j}CAT(0:\gamma_j,\mu_0) = 0 \ \forall \ j. \tag{152}$$

$$\aleph^{AT}_{j}(0) = \mu_0\gamma^{AT}_{j}\, b^{AT}_{j}\, SAT(z_\omega:\gamma_j,\mu_0). \tag{153}$$

$$\Re^{AT}_{j}(z_\omega) = \mu_0\gamma^{AT}_{j}a^{AT}_{j}CAT(z_\omega:\gamma_j,\mu_0). \tag{154}$$

$$\aleph^{AT}_{j}(z_\omega) = \mu_0\gamma^{AT}_{j}\, b^{AT}_{j}\, SAT(0:\gamma_j,\mu_0) = 0 \ \forall \ j. \tag{155}$$

3.2. Complete solution

We can now write down the complete solution in the following form.

$$I_{AT+}(z) = RE^{AT}_{+}(z) + CO^{AT}_{+}(z) + I^{p}_{AT}(+;z); \tag{156}$$

$$I_{AT-}(z) = RE^{AT}_{-}(z) + CO^{AT}_{-}(z) + I^{p}_{AT}(-;z). \tag{157}$$

$$I_{OC+}(z) = RE^{OC}_{+}(z) + CO^{OC}_{+}(z) + I^{p}_{OC}(+;z); \tag{158}$$

$$I_{OC-}(z) = RE^{OC}_{-}(z) + CO^{OC}_{-}(z) + I^{p}_{OC}(-;z). \tag{159}$$

4. BOUNDARY CONDITIONS FOR FLAT OCEAN SURFACE

Now we shall recall the boundary conditions with slight modifications for flat ocean surface.

$$I_{AT}(0,-\mu) = f(-\mu); \tag{160}$$

$$I_{AT}(z_\omega,\mu) = R(-\mu,n)I_{AT}(z_\omega,-\mu) + T(\mu,n)\frac{I_{OC}(z_\omega,\mu)}{n^2}; \tag{161}$$

$$\frac{I_{OC}(z_\omega,-\mu)}{n^2} = R(\mu,n)\frac{I_{OC}(z_\omega,\mu)}{n^2} + T(-\mu,n)I_{AT}(z_\omega,-\mu); \tag{162}$$

$$I_{OC}(z_1,\mu) = g(\mu). \tag{163}$$

$$I^{p}_{AT}(-;0) = \Delta\sum_{j=1}^{Nr}[\Re^{AT}_{j}(0)\mathbf{H}^{AT}_{-}(\gamma_j) + \aleph^{AT}_{j}(0)\mathbf{H}^{AT}_{+}(\gamma_j)] + \Delta\sum_{j=1}^{Nc} AZ^{AT}_{-}(0,\gamma^{AT}_{j}) + BZ^{AT}_{+}(0,\gamma^{AT}_{j})$$

$$= \Delta\sum_{j=1}^{Nr}[\aleph^{AT}_{j}(0)\mathbf{H}^{AT}_{+}(\gamma_j)] + \Delta\sum_{j=1}^{Nc} [\Re^{AT*}_{j}(0)\mathbf{H}^{AT*}_{-}(\gamma^{AT}_{j}) + \aleph^{AT}_{j}(0)\mathbf{H}^{AT}_{+}(\gamma^{AT}_{j}) + \aleph^{AT*}_{j}(0)\mathbf{H}^{AT*}_{+}(\gamma^{AT}_{j})] = \Xi_1. \tag{164}$$

$$R(-)\ [I^{p}_{AT}(-;z_{\omega})]-\ T(+)\ [I^{p}_{OC}(+;z_{\omega})]-\ I^{p}_{AT}(+;z_{\omega})$$

$$=R(-)[\Delta\sum_{j=1}^{Nr}[\Re^{AT}_{j}(z_{\omega})H^{AT}_{-}(\gamma_{j})+\aleph^{AT}_{j}(z_{\omega})H^{AT}_{+}(\gamma_{j})]+\Delta\sum_{j=1}^{Nc}\ [AZ^{AT}_{-}(z_{\omega},\gamma^{AT}_{j})+\ BZ^{AT}_{+}(z_{\omega},\lambda^{AT/OC}_{j})]]$$

$$-T(+)[\sum_{j=1}^{Nr}[\Re^{OC}_{j}(z_{\omega})H^{OC}_{+}(\gamma_{j})+\aleph^{OC}_{j}(z_{\omega})H^{OC}_{-}(\gamma_{j})]+\sum_{j=1}^{Nc}\ [AZ^{OC}_{+}(z_{\omega},\gamma^{OC}_{j})+\ \ BZ^{OC}_{-}(z_{\omega},\gamma^{OC}_{j})]]$$

$$-\sum_{j=1}^{Nr}[\Re^{AT}_{j}(z_{\omega})H^{AT}_{+}(\gamma_{j})+\aleph^{AT}_{j}(z_{\omega})H^{AT}_{-}(\gamma_{j})]+\sum_{j=1}^{Nc}\ AZ^{AT}_{+}(z_{\omega},\gamma^{AT}_{j})+BZ^{AT}_{-}(z_{\omega},\gamma^{AT}_{j})=\Xi_{2}. \qquad (165)$$

$$\frac{1}{n^{2}}R(+)\ [I^{p}_{OC}(+;z_{\omega})]+\ T(-)\ [I^{p}_{AT}(-;z_{\omega})]\ -\frac{1}{n^{2}}I^{p}_{AT}(-;z_{\omega})$$

$$=\frac{1}{n^{2}}R(+)\ \ [\sum_{j=1}^{Nr}[\Re^{OC}_{j}(z)H^{OC}_{+}(\gamma_{j})+\aleph^{OC}_{j}(z)H^{OC}_{-}(\gamma_{j})]+\sum_{j=1}^{Nc}\ [AZ^{OC}_{+}(z,\gamma^{OC}_{j})+BZ^{OC}_{-}(z,\lambda^{OC}_{j})]]$$

$$+T(-)[\Delta\sum_{j=1}^{Nr}[\Re^{AT}_{j}(z_{\omega})H^{AT}_{-}(\gamma_{j})+\aleph^{AT}_{j}(z_{\omega})H^{AT}_{+}(\gamma_{j})]+\Delta\sum_{j=1}^{Nc}\ [AZ^{AT}_{-}(z,\gamma^{AT}_{j})+\ BZ^{AT}_{+}(z,\lambda^{AT/OC}_{j})]]$$

$$-\frac{1}{n^{2}}\sum_{j=1}^{Nr}[\Re^{AT}_{j}(z)H^{AT}_{-}(\gamma_{j})+\aleph^{AT}_{j}(z)H^{AT}_{+}(\gamma_{j})]+\sum_{j=1}^{Nc}\ AZ^{AT}_{-}(z,\gamma^{AT}_{j})+BZ^{AT}_{+}(z,\gamma^{AT}_{j})=\Xi_{3}. \qquad (166)$$

$$[I^{p}_{OC}(+;z_{1})]=[\sum_{j=1}^{Nr}[\Re^{OC}_{j}(z)H^{OC}_{+}(\gamma_{j})+\aleph^{OC}_{j}(z)H^{OC}_{-}(\gamma_{j})]+\sum_{j=1}^{Nc}\ [AZ^{OC}_{+}(z_{1},\gamma^{OC}_{j})+BZ^{OC}_{-}(z,\lambda^{OC}_{j})]]=\Xi_{4}. \qquad (167)$$

With this assumption we conclude that we can at least find all the unknown coefficients once we solve the above set of linear equations. Hence our essential requirement for the complete solutions of the problem is at hand. But these first versions of

solutions are not general since they are only for discrete directions. We shall now develop general solutions for any desired angle with the help of discrete solutions in the next sub section.

5. POST PROCESSING PROCEDURE

To get the solutions for any desired direction we shall adopt post processing procedure [11] by substituting the discrete solutions in the right hand side of the discretized equation of transfer in both the media. We recall our equation of transfer (1-2).

$$\mu\frac{d}{dz}I_{AT/OC}(z,\mu)+I_{AT/OC}(z,\mu)=\frac{\omega^{AT/OC}(z)}{2}\sum_{J=S}^{M}\mathbf{P}_{J=S}^{M}(\mu)\mathbf{B}_{J}^{AT/OC}\sum_{\alpha=1}^{N}\omega_\alpha I_{j,\alpha}^{AT/OC}(z)+S_{AT/OC}(z,\mu) \tag{168}$$

$$I_{j,\alpha}^{AT/OC}(z)=\mathbf{P}_j^m(\mu_\alpha)I_{AT/OC}(z,\mu_\alpha)+\mathbf{P}_j^m(-\mu_\alpha)I_{AT/OC}(z,-\mu_\alpha), \tag{169}$$

Integrating these two equations we get solutions for any optical depth in the atmosphere and ocean from the following four equations depending on the direction.

$$I_{AT}(z(at),+\mu)=I^{AT}(z_\omega,\mu)\exp\left(-\frac{z_\omega-z(at)}{\mu}\right)+\int_{z(at)}^{z_\omega}RSAT(x,\mu)\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}, \tag{170}$$

$$I_{AT}(z(at),-\mu)=I^{AT}(0,-\mu)\exp\left(-\frac{z(at)}{\mu}\right)+\int_{0}^{z(at)}RSAT(x,-\mu)\exp\left(-\frac{z(at)-x}{\mu}\right)\frac{dx}{\mu}, \tag{171}$$

$$I_{OC}(z(oc),+\mu)=I^{OC}(z_1,\mu)\exp\left(-\frac{z_1-z(oc)}{\mu}\right)+\int_{z(oc)}^{z_1}RSOC(x,\mu)\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}, \tag{172}$$

$$I_{OC}(z(oc),-\mu)=I^{OC}(z_\omega,-\mu)\exp\left(-\frac{z(oc)}{\mu}\right)+\int_{z_\omega}^{z(oc)}RSOC(x,-\mu)\exp\left(-\frac{z(oc)-x}{\mu}\right)\frac{dx}{\mu}. \tag{173}$$

$$RSAT(z,\mu)=\frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{P}_{J=S}^{M}(\mu)\sum_{\alpha=1}^{N}\omega_\alpha I_{j,\alpha}^{AT}(z)+S_{AT}(z,\mu), \tag{174}$$

$$RSOC(z,\mu)=\frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{P}_{J=S}^{M}(\mu)\sum_{\alpha=1}^{N}\omega_\alpha I_{j,\alpha}^{OC}(z)+S_{OC}(z,\mu), \tag{175}$$

$$\begin{aligned}I_{AT}(0,\mu)=\;&R(-\mu,n)I^{AT}(0,-\mu)\exp(-\frac{z_\omega}{\mu})\exp(-\frac{z_\omega}{\mu})+R(-\mu,n)\exp(-\frac{z_\omega}{\mu})\int_0^{z_\omega}RSAT(x,-\mu)\exp(-\frac{z_\omega-x}{\mu})\frac{dx}{\mu}\\&+\frac{T(\mu,n)}{n^2}\exp(-\frac{z_\omega}{\mu})I^{OC}(z_1,\mu)\exp(-\frac{z_1-z_\omega}{\mu})+\frac{T(\mu,n)}{n^2}\exp(-\frac{z_\omega}{\mu})\int_{z_\omega}^{z_1}RSOC(x,\mu)\exp(-\frac{x-z_\omega}{\mu})\frac{dx}{\mu}+\int_0^{z_\omega}RSAT(x,\mu)\exp(-\frac{x}{\mu})\frac{dx}{\mu}.\end{aligned} \tag{176}$$

All the terms on the right hand side of are known provided the integrals involving are properly evaluated. In expressions

(170-173) the term inside the integral sign on the right hand side, requires attention. It involves $I_{AT}(z,\pm\mu_\alpha)$ and $I_{OC}(z,\pm\mu_\alpha)$ which we have found in our previous analysis vide equations (156-159).

6. NUMERICAL RESULTS

The following table represents the values of individual stokes parameters for directions given in the first column. The source functions for both atmosphere and ocean are calculated from equations (28-35) .Specular reflection and transmission functions are evaluated using equations (13-14) We have chosen initial direction of incident beam (μ_0) as 0.74176. The optical depth corresponding to the bottom of the ocean surface is set as Z1=2.0. The interface optical depth is set at Zwas1.0.

Table 1

Z(OC)=1.1

Direction(mu1)	L	Q	U	V
0.043633	3.9736	-0.45999	7.007	0.40979
0.07854	4.0324	-0.45999	7.2183	0.41113
0.11345	4.1354	-0.50868	7.4286	0.41501
0.14835	4.2339	-0.54337	7.6068	0.4173
0.18326	4.3054	-0.57454	7.7131	0.41521
0.21817	4.3379	-0.59883	7.7253	0.40729
0.25307	4.3292	-0.61483	7.641	0.39357
0.28798	4.2832	-0.62245	7.4705	0.37481
0.32289	4.206	-0.62223	7.2286	0.35208
0.3927	3.9849	-0.60159	6.5916	0.29856
0.42761	3.8523	-0.5828	6.2222	0.2693
0.46251	3.7112	-0.55932	5.8327	0.2391
0.53233	3.4162	-0.50053	5.0219	0.17724
0.60214	3.1194	-0.42882	4.2024	0.11469
0.63705	2.9741	-0.38889	3.7978	0.083393
0.67195	2.8322	-0.34648	3.3996	0.052125
0.70686	2.6954	-0.30118	3.0091	0.02087
0.74176	2.5441	-0.2454	2.6281	-0.010292
0.77667	2.4288	-0.21483	2.552	-0.041794
0.81158	2.3024	-0.16345	1.8935	-0.073176

Table 2

Z(OC)=1.3

Direction(mu1)	L	Q	U	V
0.043633	11.622	-3.2546	-9.2975	-0.66569
0.07854	5.0627	-0.78343	-9.564	-0.71276
0.11345	4.9832	-0.74425	-9.7951	-0.75858
0.14835	5.0282	-0.76437	-9.9092	-0.7965
0.18326	5.0323	-0.78273	-9.8658	-0.82258
0.21817	4.9778	-0.79077	-9.6713	-0.83578
0.25307	4.8724	-0.78793	-9.3553	-0.83715
0.28798	4.7289	-0.77563	-8.9503	-0.82844
0.32289	4.5592	-0.75559	-8.4851	-0.81137
0.3927	4.1785	-0.69821	-7.4582	-0.75806
0.42761	3.9802	-0.66309	-6.9253	-0.72408
0.46251	3.7824	-0.62476	-6.3924	-0.68636
0.53233	3.398	-0.54055	-5.3478	-0.60206
0.60214	3.0384	-0.4486	-4.3528	-0.50878
0.63705	2.8695	-0.40025	-3.8778	-0.45952
0.67195	2.7086	-0.35036	-3.4189	-0.4088
0.70686	2.5571	-0.29825	-2.9759	-0.35676
0.74176	2.3872	-0.23409	-2.5485	-0.30342
0.77667	2.2685	-0.20431	-2.1387	-0.24941
0.81158	2.1352	-0.14805	-1.7427	-0.19411

7. CONCLUSION

In this paper, we achieved an efficient numerical technique based on analytical developments from the exact expressions of the radiative source and flux. The source and flux are spatially discretized without any ray effect, which leads to a set of quasi-analytical coefficients depending only on the geometry of the system, calculated once with a Gauss quadrature and stored. These coefficients completely describe the radiative field which does not need to be iteratively re-evaluated in the case of radiation coupled with other heat transfers, contrarily to other methods such as DOM, finite volumes or finite elements method. Furthermore, this description easily handles the boundary conditions on the surfaces of the two cylinders.

The numerical results for the temperature field and the radiative flux at radiative equilibrium have been compared to other methods and are extremely reliable for all tested optical depths. The method produces smooth temperature and flux fields without any oscillatory errors in a satisfactory computation time. The extension of this present work to obtain the complete radiative field inside scattering media with a linearly varying phase function is underway when the two boundaries are diffusely reflecting. A similar study is also examined for purely specularly reflecting surfaces.

8. ACKNOWLEDGEMENS

Support for this work was provided by the Mathematics Department at the University of North Bengal.

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ISSN 2278-7763