Stress Functions in a Thin Annular Disc Due To Partially Distributed Heat Supply

Sunil D. Bagde[1] and N.W.Khobragade[2]

[1]Research scholar and [2]Professor, Post Graduate Teaching Department of Mathematics MJP Educational Campus, RTM Nagpur University, Nagpur – 440 033, India.
E-mail:sunilkumarbagde@rediffmail.com

ABSTRACT- This paper concerned with stress functions in thin annular disc due to partially distributed heat supply to determine the temperature, displacement function and stress functions with the help of finite Fourier cosine transform, Marchi-Zgrablich transform and Laplace transform techniques.

Keywords : Inverse thermoelastic problem, Thin annular disc, Fourier cosine transform, Marchi-Zgrablich transform and Laplace transform.

1 INTRODUCTION

Nowacki [1] has determined steady-state thermal stresses in a circular plate subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and the circular edge respectively.

In this paper, an attempt has been made to determine the temperature (in heating and cooling process), displacement and stress functions of the annular disc. With boundary conditions. Using finite Fourier sine transform, Marchi- Zgrablich transform and Laplace transform techniques. The results are expressed in the form of infinite series and depicted graphically.

2. STATEMENT OF THE PROBLEM (HEATING PROCESS)

Consider a thin annular disc occupying the space, $\xi_0 \le r \le \xi_1 \; 0 \le z \le h$. The initial temperature of the disc is the same as the temperature of the surrounding medium which is kept constant for the time $t = 0$ to $t = t_0$ the disc is subjected to a partially distributed and axi-symmetric heat supply $\left(-P_0 \, F(r,t)/\lambda\right)$ from the interior point (r, ξ). After that, the heat supply is removed and disc is cooled by the surrounding medium.
The differential equation governing the displacement function $U(r, z, t)$ as [1] is

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = (1+\nu)\alpha_t T \tag{1}$$

with $V_r = 0$ at $r = \xi_0$ and $r = \xi_1$ (2)

where ν and α_t are Poisson's ratio and linear coefficient of thermal expansion of the material of the disc respectively and $T(r, z, t)$ is the heating temperature of the disc at time t satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial T}{\partial z} \tag{3}$$

subject to the initial condition

$$[T(r, z, t)]_{t=0} = 0 \text{, for all } \quad \xi_0 \le r \le \xi_1 \tag{4}$$

The boundary conditions

$$\left\{T(r, z, t) + k_1 \frac{\partial T(r, z, t)}{\partial r}\right\}_{r=\xi_0} = 0 \text{, for all z , t > 0} \tag{5}$$

$$\left\{T(r, z, t) + k_2 \frac{\partial T(r, z, t)}{\partial r}\right\}_{r=\xi_1} = 0 \text{, for all z , t > 0} \tag{6}$$

$$\left[\frac{\partial T(r, z, t)}{\partial z}\right]_{z=0} = 0 \text{, For all r,t>0} \tag{7}$$

$$\left[\frac{\partial T(r, z, t)}{\partial z}\right]_{z=h} = g(r,t) \text{, (unknown) for all r , t > 0} \tag{8}$$

The interior condition

$$\left[\frac{\partial T(r, z, t)}{\partial z}\right]_{z=\xi} = -\frac{P_0}{\lambda} f(r,t) \text{, (known)} \tag{9}$$

where k and λ are the thermal diffusivity and conductivity of the material of the disc respectively, k_1 and k_2 are radiation constants on the curved surfaces of the disc respectively. The stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = -2\mu \frac{1}{r}\frac{\partial V}{\partial r} \tag{10}$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 U}{\partial r^2} \tag{11}$$

Equations (1) to (11) constitute the mathematical formulation of the problem under consideration.

3. DETERMINATION OF HEATING TEMPERATURE

On applying Marchi-Zgrablich transform defined in [2] to the equations (3), (4), (7) to (9) and using (5), (6), one obtains

$$\frac{d^2T^*}{dz^2} - {\mu_n}^2 T^* = \frac{1}{k}\frac{dT^*}{dt} \tag{12}$$

$$[T^*(n, z, t)]_{t=0} = 0\,, \tag{13}$$

$$\left[\frac{dT^*(n, z, t)}{dz}\right]_{z=0} = 0\,, \tag{14}$$

$$\left[\frac{dT^*(n, z, t)}{dz}\right]_{z=h} = g^*(n,t)\,, \tag{15}$$

$$\left[\frac{dT^*(n, z, t)}{dz}\right]_{z=\xi} = -\frac{P_0}{\lambda} F^*(n,t) \tag{16}$$

where T^* is the Marchi-Zgrablich transform of T and n is the Marchi-Zgrablich transform parameter.
Applying Laplace transform defined in [3] to the equation (12), (14) to (16) and using (13), one obtains

$$\frac{d^2\overline{T}}{dz^2} - q^2\overline{T} = 0 \tag{17}$$

$$\left[\frac{d\overline{T}(n, z, s)}{dz}\right]_{z=0} = 0\,, \tag{18}$$

$$\left[\frac{d\overline{T}(n, z, s)}{dz}\right]_{z=h} = \overline{g}(n,s)\,, \tag{19}$$

$$\left[\frac{d\overline{T}(n, z, s)}{dz}\right]_{z=\xi} = -\frac{P_0}{\lambda} \overline{f}(n, s) \tag{20}$$

where $q^2 = {\mu_n}^2 + \frac{s}{k}$, $\overline{T}$ is the Laplace transform of $\overline{T}$ and s is the Laplace transform parameter.
Equation (17) is a second order differential equation whose solution is given by

$$\overline{T}(n, z, s) = Ae^{qz} + Be^{-qz} \tag{21}$$

where A, B are arbitrary constants.
Using (18) and (20) in (21) we obtain

$$A = -\left(\frac{1}{q(e^{q\xi} - e^{-q\xi})}\right)\left(\frac{P_0\overline{F}(n, s)}{\lambda}\right),$$

$$B = -\left(\frac{1}{q(e^{q\xi} - e^{-q\xi})}\right)\left(\frac{P_0\overline{F}(n, s)}{\lambda}\right)$$

Substituting the values of A and B in equation (21), one obtains

$$\overline{T}(n, z, s) = -\left(\frac{P_0\overline{F}(n, s)}{\lambda}\right)\left[\frac{\cosh(qz)}{q\sinh(q\xi)}\right] \tag{22}$$

Applying inversion of Laplace transform and finite Marchi-Zgrablich transform to the equation (22), one obtain the expression for temperature gradient and unknown temperature as

$$T(r, z, t) = \sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\frac{2kP_0}{\lambda\xi}\left\{\sum_{m=0}^{\infty}\frac{\cos(\lambda_m z)}{\cos(\lambda_m \xi)}\right\}\int_0^t \overline{f}(n, t')e^{-k(\mu_n^2+\lambda_m^2)(t-t')}dt'\right\} \times S_0(k_1, k_2, \mu_n r) \tag{23}$$

$$g(r,\ t) = \sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\frac{2kP_0}{\lambda\xi}\left\{\sum_{m=0}^{\infty}\frac{\lambda_m\sin(\lambda_m h)}{\cos(\lambda_m \xi)}\right\}\int_0^t \overline{f}(n, t')e^{-k(\mu_n^2+\lambda_m^2)(t-t')}dt'\right\} \times S_0(k_1, k_2, \mu_n r) \tag{24}$$

where m, n are positive integers, $S_0(k_1, k_2, \mu_n r)$ is kernel of the transform, μ_n are positive roots of $S_0(k_1, k_2, \mu_n r) = 0$, λ_m are positive roots of $\cos(\lambda_m \xi) = 0$,

$$\overline{F}(n,t) = \int_a^b r\ f(r,t)\ S_0(k_1, k_2, \mu_n r)\ dr\,,$$

$$C_n = \int_a^b r\{S_0(k_1, k_2, \mu_n r)\}^2\, dr\,.$$

4. STATEMENT OF THE PROBLEM (COOLING PROCESS)

The temperature change $T'(r, z, t)$ for the cooling process satisfies the equation

$$\frac{\partial^2\overline{T}}{\partial r^2} + \frac{1}{r}\frac{\partial\overline{T}}{\partial r} + \frac{\partial^2\overline{T}}{\partial z^2} = \frac{1}{k}\frac{\partial\overline{T}}{\partial t} \tag{25}$$

$$\{\overline{T}(r, z, t)\}_{t=t_0} = T(r, z, t_0) \tag{26}$$

The boundary conditions

$$\left\{\overline{T}(r, z, t) + k_1\frac{\partial\overline{T}(r, z, t)}{\partial r}\right\}_{r=\xi_0} = 0 \tag{27}$$

$$\left\{\overline{T}(r, z, t) + k_2\frac{\partial\overline{T}(r, z, t)}{\partial r}\right\}_{r=\xi_1} = 0 \tag{28}$$

$$\left[\frac{\partial\overline{T}(r, z, t)}{\partial z}\right]_{z=0} = 0 \tag{29}$$

$$\left[\frac{\partial\overline{T}(r, z, t)}{\partial z}\right]_{z=h} = 0 \tag{30}$$

Where $T(r, z, t)$ is the heating temperature of the disc at time t satisfying the differential equation (3)

5. DETERMINATION OF TEMPERATURE OF COOLING PROCESS

On applying Marchi-Zgrablich transform [2] to the equations (25), (26), (29),(30) using (27) and (28), one obtains

$$\frac{d^2\overline{\overline{T}}}{dr^2} - {\mu_n}^2 \overline{\overline{T}} = \frac{1}{k}\frac{d\overline{\overline{T}}}{dt} \tag{31}$$

$$\{\overline{\overline{T}}(n, z, t)\}_{t=t_0} = \overline{\overline{T}}(n, z, t_0) \tag{32}$$

$$\left[\frac{d\overline{\overline{T}}(n, z, t)}{dz}\right]_{z=0} = 0 \tag{33}$$

$$\left[\frac{d\overline{\overline{T}}(n, z, t)}{dz}\right]_{z=h} = 0 \tag{34}$$

Where $\overline{\overline{T}}$ is the Marchi-Zgrablich transform of $\overline{T}$ and n is the Marchi-Zgrablich transform parameter.
Applying finite Fourier Cosine Transform to the equations (31),(32) and using (33),(34), one obtains

$$\frac{d\overline{\overline{\overline{T}}}}{dt} + k({\mu_n}^2 + p^2)\overline{\overline{T}} = 0 \tag{35}$$

$$\left\{\overline{\overline{T}}(n, m, t)\right\}_{t=t_0} = \overline{\overline{T}}(n, m, t_0) \tag{36}$$

Where $\overline{\overline{T}}$ is the Fourier Cosine Transform of $\overline{T}$ and m is the Fourier cosine transform parameter.
Applying (5.6) in (5.5), one obtains

$$\overline{\overline{T}}(n,m,t) = \overline{\overline{T}}(n,m,t_o)\, e^{-k(\mu_n^2+p^2)(t-t_0)} \tag{37}$$

Applying inverse finite Fourier cosine transform and finite March-Zgrablich transform to the equations (5.5), one obtains

$$\overline{T}(r, z, t)$$
$$= \sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\frac{2}{h}\sum_{m=1}^{\infty}\left[\frac{2kP_0}{\lambda\xi}\left\{\sum_{l=0}^{\infty}\frac{\cos(\lambda_l z)}{\cos(\lambda_l \xi)}\right\}\int_0^{t_0}\overline{f}(n, t')e^{-k(\mu_n^2+\lambda_l^2)(t-t')}\, dt'\, e^{-k(\mu_n^2+p^2)(t-t_0)}\right.\right.$$
$$\times S_0(k_1, k_2, \mu_n r) \tag{38}$$

Where $p = \dfrac{m\pi}{h}$, $\lambda_l = \dfrac{l\pi}{h}$ and m, n, l are positive integers.

6. DETERMINATION OF DISPLACEMENT FUNCTION

Substituting the value of $T(r, z, t)$ from (23) in (1), one obtains the thermoelastic displacement function $U(r, z, t)$ as

$$U(r, z, t)$$
$$= -\frac{2(1+\nu)kP_0 a_t}{\lambda\xi}\sum_{n=1}^{\infty}\frac{1}{C_n\mu_n^2}\left\{\sum_{m=0}^{\infty}\frac{\cos(\lambda_m z)}{\cos(\lambda_m \xi)}\right\}\left\{\int_0^t \overline{F}(n, t')\, e^{-k(\mu_n^2+{\lambda_m}^2)(t-t')}dt'\right\}$$
$$\times S_0(k_1, k_2, \mu_n r) \tag{39}$$

7. DETERMINATION OF STRESS FUNCTIONS

Substituting the value of *U(r,z,t)* from (39) in (10) and (11), one obtains the stress functions as

$$\sigma_{rr}$$
$$= \frac{4\mu(1+\nu)kP_0 a_t}{r\lambda\xi}\sum_{n=1}^{\infty}\frac{1}{C_n\mu_n}\left\{\sum_{m=0}^{\infty}\frac{\cos(\lambda_m z)}{\cos(\lambda_m \xi)}\right\}\left\{\int_0^t \overline{f}(n, t')\, e^{-k(\mu_n^2+{\lambda_m}^2)(t-t')}dt'\right\}$$
$$\times S_0'(k_1, k_2, \mu_n r) \tag{40}$$

$$\sigma_{\theta\theta}$$
$$= \frac{4\mu(1+\nu)kP_0 a_t}{\lambda\xi}\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=0}^{\infty}\frac{\cos(\lambda_m z)}{\cos(\lambda_m \xi)}\right\}\left\{\int_0^t \overline{f}(n, t')\, e^{-k(\mu_n^2+{\lambda_m}^2)(t-t')}dt'\right\}$$
$$\times S_0'(k_1, k_2, \mu_n r) \tag{41}$$

9. CONCLUSION

The problem, we have investigated the temperature, displacement and stress functions at any point of the disc, when the interior heat flux and the other boundary conditions are known, with the aid of finite Fourier sine transform, Laplace transform and Marchi-Zgrablich transform techniques.
The expression useful in Engineering problems particularly in the determination of the state of strain in the disc constituting the foundations of container for liquid or hot gases etc.

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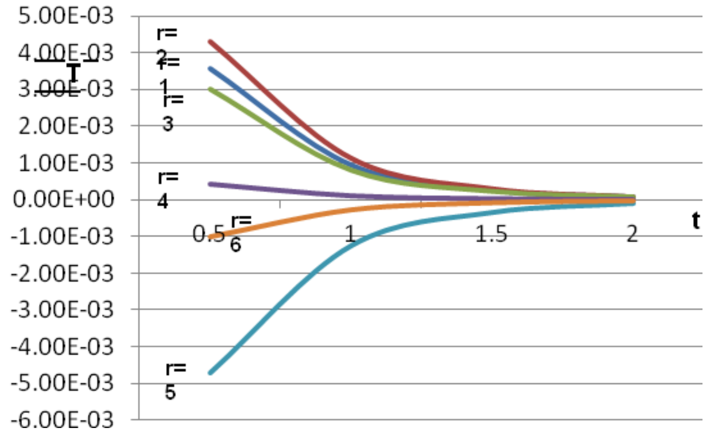

Graph $\bar{T}$ Versus t for different values of r