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Representation of Product Bases for the Rationals

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ABSTRACT. A sequence of positive rationals generates a sub group of finite index in the multiplicative positive rationals, and group product representation by the sequence need only a bounded number of terms, if and only if certain related sequences have densities uniformly bounded from below.

In 1930 Scchnirelmann introduced on the positive integers the density now named after him , and employed it to show that every integer greater than 1 could be expressed as a sum of primes ,with a uniformly bounded number of summands. This was a first step towards the conjectures of Goldbach .It was important that once positive, Schnirelmann,s density is increased by adding a sequence to itself.

Let Q^* denote the multiplicative group of positive rationals, Γ the subgroup generated by the members, a_n , of a sequence of rationals,G the quotient group Q^* / Γ .

A sequence of rationals may generate a proper sub group of Q* ,Unlike the situation for an integer sequence of positive Schnirelmann density, further multiplication will avail us naught :, Q^* is not cyclic : there is no multiplicative element corresponding to 1 .

Whether a positive integer may be expressed as a sum of elements taken from some sequence of distinct positive integers may be decided in a finite number of steps. Not surprisingly, Schnirelmann, s arguments are local in nature .However, the word problem for denumerable abelian groups has no (finite recursive) decision procedure. Since Q* is free on denumerably many generators, and factoring by Γ plays the role of adjoining relations, the nature of G cannot be determined, locally or not ,without particular knowledge of the a_n in Γ .What should this knowledge be ?

The following result throws a little light on this problem.

For each positive integer t, S_i will denote the members of Γ representable by a group product using at most t term from the sequence (a_n), repetition allowed.

For a set of rational ,E ,define the lower density

$$d(E) = \lim_{x\to\infty} \inf \ x^{-1} \sum_{n\le x, n\in E} 1 ,$$

the sum counting only positive integers. Clearly $0 \le d(E) \le 1$ and the density of a union of disjoint sets E and F is at least as large as the sum of their individual densities $.d(E)$ will be the classical density if the infimum may be removed.

γ E will denote the set of products of the elements in E by the rational γ .

Theorem

(i) G is finite and $\tau = S_t$ for some t

(ii) There is a k such that $d(m^{-1} sk) \ge c_1 > 0$ uiformly for all positive integers m .

(iii) There is a w such that $d(\gamma s_w) \ge c_2 > 0$ uniformly for all positive rationals γ .

Moreover, t can be given explicitly in terms of k and c1 , or w and c2 .

In part, proposition (i) of the theorem asserts the existence of a t , so that if a rational belongs to τ , then at most t members of the sequence (an) are needed in its product representation. Nothing is said concerning the realization of such a representation.

That (ii) implies (i) could be obtained by adapting the short argument in Elliot (2), but t would not then be computable.

Combined with results from number theory , a careful variant of the argument given here shows that any rational representable as a product of shifted primes p+1 , has such a representation with exactly 19 terms (1).

We present a cycle proof of the theorem . Theorem. That (iii) implies (ii) is clear.

Proof that (ii) implies (i)

Let the integer h exceed C^{-1} . Let m be a positive integer. The sets sk , m^{-j} sk 0 ≤ j≤ h-1 ,cannot be mutually disjoint , otherwise If γ = a b^{-1} , with integers a, b, ,and b′.

The sets $m^{-1}{}_j$ S_k , 1 ≤ j ≤ f are mutually disjoint .Their union has density at least c1f . The group G is bounded, of order at most $c^{-1}{}_j$.

That ɼ is some st costs a little more . We use the fact that if S_{r+1} $=S_r$, then ɼ = S_r ,
E- F will denote the set of elements in E but not F.
Let γ belongs to S_{r+1} − S_r , r ≥ 3k ,If γ = a b^{-1} ,as before, with bz in S2k ,then Sr+1+2k − Sr−2k contain the integer abz−1 .
Moreover, if Sr+1 − Sr contains a rational, then so do each of Sj+1 − Sj 3k ≤ j ≤ r .We can find an integer in each set Sj+1+2k − Sj−2k , 3k ≤ j ≤ r . The set S7kw+1+2k − S7kw−2k , 1 ≤ w ≤ r ($7k^{-1}$) are also disjoint . Their union has density at least r c($7k^{-1}$) .
If r > 7k c^{-1}1 ,then we obtain a contradiction. Hence r is St for some t not exceeding 7k c^{-1}1 .
No attempt has been made to minimize the value of t .
Remark Let $\bar{S}$ J denote the set of element in ɼ representable using exactly j of the an , counted with multiplicity.

$\bar{S}$ J to be contained in $\bar{S}$ J+1 there must be a representation of 1 by an odd number of the an .Granted to the existence of such a representation . $\bar{S}$ J will be a subset of $\bar{S}$ J+m for all m≥ (an odd) y .

Assume further ,the validity of the second part of proposition(1). Then for j ≥ t+y ,

$$\bar{S}_J \subseteq ɼ = St = \bigcup_{i=1}^{t} \bar{S}_J \subseteq \bigcup_{i=1}^{t} \bar{S}_{i+(j-1)} = \bar{S}_J$$

Each member of ɼ has a product representation employing exactly j term from (an).

Proof that (i) implies (iii) That the integers belonging to any particular coset of ɼ have an asymptotic density follows from theory (3.10) 0f Ruzsa (3). More over ,since our group G is finite , at least one density is positive . By the same result every density is positive . Our situation is simpler than that of Ruzsa ,and it is pertinent to obtain reasonably explicit values for the various densities rather than that give them bounds. For completion and not to appeal to his somewhat complicated paper we give a proof .Like Ruzsa , we adapt a method of Drichlet more closely .
Let σ denote the canonical homomorphism Q* → $\frac{Q^*}{\Gamma}$.

Let the ω in G for which the sum $\sum p^{-1}$,taken over the primes for which σ(p)= ω diverges the subgroup G1 of G . In particular, $\sum q^{-1}$, taken over the primes q for which σ(p) does not lie in G1 , converges.
If a rational is not divisible by a q , then it maps into G1 .
Let Q1 denote the subgroup of positive rationals not divisible by and q.
Let θ_j ,1≤ j ≤ |Gj | denote the characters on G1 ,and gj the composition $\theta_j\sigma$ restricted to Q1 .We extend gj to Q* by setting every gj(q) =0 .We obtain in this way a multiplicative arithmetic function with values in complex unit disc .
Those integers n in Q1 .which σ takes to a particular element μ (of G1)have an asymptotic density;

$$D(\mu) = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \frac{1}{|Gj|} \sum_{j-1}^{|Gj|} gj(n)\overline{\theta j(\mu)} = \frac{1}{|Gj|} \sum_{j=1}^{|GJ|} \overline{\theta J(\mu)} \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} gj(n)$$

Here we employ a theorem of wirsing (4), that a multiplicative function which assumes only finitely many values in the complex unit disc possesses an asymptotic mean- value .
According to that same reference ,the mean –value of a typical gj is zero unless the series $\sum p^{-1}$(1−Re gj(p)) converges . If δ denotes the minimum of 1−Re p taken over the $|Gi|$-th roots of unity , p, then this series cannot converges , we may omit the condition the condition ;p and q .Then θ_j must be trivial on

every ω , and so on G1 . For the gj which extends the principal character on G1 we count those integers not divisible by any q ; an application of the sieve of Eratosthenes suffices . We arrive at the value D(μ) = $|G1|^{-1} \prod (1-- \frac{1}{q})$.

Which is independent of μ .

Each positive integer has a unique representation n1n2 ,where n1 is that part of n comprised of powers of the primes q , an empty product interpreted as 1 . Corresponding to this decomposition, for any element τ in G,

$$\sum_{\substack{n\le x \\ \sigma(n)=r}} 1 = \sum_{\lambda}$$

$$\sum_{\substack{n\le x \\ \sigma(n)=r}} 1 = \sum_{\lambda} \sum_{\substack{n1\le x \\ \sigma(n1)=\lambda}} \sum_{\substack{n2\le \frac{x}{n1} \\ \sigma(n2)=\tau\lambda^{-1}}} 1$$

Where λ runs over the elements of G. Since n2 belongs to Q1 ,the innersum will be empty unless $\tau\lambda^{-1}$ belongs to G1 ; We may then estimate it in terms of D ($\tau\lambda^{-1}$). In this way we

$$\Delta(\tau) = \sum_{\lambda\in\tau G1} \sum_{\sigma(n1)=\lambda} \frac{1}{n1|G1|} \prod(1-\frac{1}{q})$$

This density is positive and does not exceed $|G1|^{-1}$.

We continue with the proof of our theorem . Until now we have employed only the first part of (i) . Appealing to the second part , ther are only finitely many distinct cosets $\gamma St = \gamma \Gamma$, γ rational .For each of these

$$d(\gamma St) = \Delta(\gamma \pmod{\Gamma}) > 0.$$

Proposition (iii) is valid with w= t and C2 the minimum of $\Delta(\tau)$ taken over the elements τ in G.

The theorem is proved.

Remarks When Γ is generated by the shifted primes, application of Drichlet ' theorem on primes in arithmetic progression together with the sieve of Eratosthenes shows that G1 coincides withG ; of (1,Lemma7) Every coset of Γ then has asymptotic density $|G1|^{-1}$.

It is clear that one may replace the lower density in the theorem by any other density that is atleast additive on disjoint sets and coincides with the standard asymptotic density when that exist.

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