

RECIPROCAL CONTINUITY AND COMMON FIXED POINT FOR TWO PAIRS OF SELF-MAPS SATISFYING A GENERALIZED INEQUALITY

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ABSTRACT

We obtain a common fixed point for two pairs of self-maps on a complete metric space, one of which is reciprocally continuous and compatible, while the other weakly compatible, where all the four maps satisfy a generalized inequality. Our result is a significant generalization of that of Singh and Mishra.

Keywords : Complete Metric Space; Reciprocal Continuity; Compatible and Weakly Compatible Self-maps; Common Fixed Point

1 INTRODUCTION

Let (X, d) be a metric space with metric d . If $x \in X$ and f a self-map on X , we write fx for the f -image of x , $f(X)$ for the range of f and Sf for the composition of self-maps S and f . Gerald Jungck [2] defined a pair of self-maps S and f on X to be compatible (or *asymptotically commuting*) if

$$\lim_{n \rightarrow \infty} d(Sfx_n, fSx_n) = 0 \quad (1.1)$$

whenever $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} Sx_n = t \text{ for some } t \in X. \quad (1.2)$$

Now taking $x_n = x$ for all n , the compatibility implies that $Sfx = fSx$ whenever $fx = Sx$. Self-maps which commute at their coincidence points are known to form a *weakly compatible* or *coincidentally commuting* pair [3]. Weakly compatible maps are also called partially commuting [5].

Self-maps S and f on X are *reciprocally continuous*, abbreviated as *rc* if for any $\langle x_n \rangle_{n=1}^{\infty} \subset X$ with the choice (1.2), we have

$$\lim_{n \rightarrow \infty} fSx_n = ft \text{ and } \lim_{n \rightarrow \infty} Sfx_n = St \text{ for some } t \in X.$$

Any pair of continuous maps will obviously be a reciprocally continuous one. However a pair of maps S and f may be reciprocally continuous without both the maps being continuous [4].

We need the following basic idea from [1], which is stated without proof:

Lemma 1. A contractive sequence $\langle p_n \rangle_{n=1}^{\infty} \subset X$ such that

$$d(p_n, p_{n-1}) \leq q d(p_{n-1}, p_{n-2}) \text{ for } n \geq 1$$

is a Cauchy sequence, where $0 < q < 1$.

We also use the continuity of the metric d and the property:

$$\max\{a, b, \frac{a+b}{2}\} \leq \max\{a, b\}. \quad (1.3)$$

2 MAIN RESULT

We now prove the following:

Theorem 1. Let f, g, S and T be self-maps on a complete metric space X satisfying

$$S(X) \subset g(X), T(X) \subset f(X) \quad (2.1)$$

and the inequality

$$\begin{aligned} [1 + \beta d(fx, gy)]d(Sx, Ty) \\ \leq \alpha \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), \\ \frac{1}{2}[d(fx, Ty) + d(Sx, gy)] \\ + \beta[d(fx, Sx)d(gy, Ty) + d(fx, Ty)d(Sx, gy)]\} \\ \text{for all } x, y \in X, \end{aligned} \quad (2.2)$$

where $0 \leq \alpha < 1$, $\beta \geq 0$. Suppose that either of the pairs (S, f) and (T, g) is *rc* and compatible, while the other weakly compatible. Then all the four maps f, g, S and T will have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. In view of the inclusions (2.1), we can choose points x_1, x_2, \dots in X inductively such that

$$\begin{aligned} Sx_{2n-2} = gx_{2n-1} = y_{2n-1}, Tx_{2n-1} = fx_{2n} = y_{2n} \\ \text{for } n = 1, 2, \dots \end{aligned} \quad (2.3)$$

Write $t_n = d(y_n, y_{n+1})$, $n \geq 1$. We first show that

$$t_n \leq t_{n-1} \text{ for all } n \geq 2. \quad (2.4)$$

In fact, taking $x = x_{2n-2}$ and $y = x_{2n-1}$ in (2.2) and using (2.3), we get

$$\begin{aligned}
 & [1 + \beta d(fx_{2n-2}, gx_{2n-1})]d(Sx_{2n-2}, Tx_{2n-1}) \\
 & \leq \alpha \max\{\beta d(fx_{2n-2}, gx_{2n-1}), d(fx_{2n-2}, Sx_{2n-2}), d(gx_{2n-1}, Tx_{2n-1}), \\
 & \quad \frac{1}{2}[d(fx_{2n-2}, Tx_{2n-1}) + d(Sx_{2n-2}, gx_{2n-1})] \\
 & \quad + \beta[d(fx_{2n-2}, Sx_{2n-2})d(gx_{2n-1}, Tx_{2n-1}) \\
 & \quad + d(fx_{2n-2}, Tx_{2n-1})d(Sx_{2n-2}, gx_{2n-1})]\}
 \end{aligned}$$

or

$$\begin{aligned}
 & [1 + \beta d(y_{2n-2}, y_{2n-1})]d(y_{2n-1}, y_{2n}) \\
 & \leq \alpha \max\{\beta d(y_{2n-2}, y_{2n-1}), d(y_{2n-2}, y_{2n-1}), d(y_{2n-2}, y_{2n}), \\
 & \quad \frac{1}{2}[d(fx_{2n-2}, Tx_{2n-1}) + d(Sx_{2n-2}, gx_{2n-1})] \\
 & \quad + \beta d(y_{2n-2}, y_{2n-1})d(y_{2n-1}, y_{2n})\}.
 \end{aligned}$$

Cancelling the common term both sides, and using (1.3), this implies

$$t_{2n-1} \leq \alpha \max\{t_{2n-2}, t_{2n-1}\}. \tag{2.5}$$

On the other hand, taking $x = x_{2n}$ and $y = x_{2n-1}$ in (2.2) and proceeding as above, we see that

$$t_{2n} \leq \alpha \max\{t_{2n-1}, t_{2n}\}. \tag{2.6}$$

Combining (2.5) and (2.6), it follows that

$$t_n \leq \alpha \max\{t_{n-1}, t_n\} \text{ for all } n \geq 2. \tag{2.7}$$

If $t_k > t_{k-1}$ for some $k \geq 2$, then $t_k > 0$ and $\max\{t_{k-1}, t_k\} = t_k$ so that (2.7) would give a contradiction that $t_k \leq \alpha t_k < t_k$, since $\alpha < 1$. Thus $t_n \leq t_{n-1}$ for all n . Using this in (2.7), we obtain (2.4) and hence by Lemma 1, it follows that $\langle y_n \rangle_{n=1}^\infty$ is a Cauchy sequence in X . Since X is complete, we can find a point $z \in X$ with $\lim_{n \rightarrow \infty} y_n = z$ so that

$$\lim_{n \rightarrow \infty} fx_{2n} = \lim_{n \rightarrow \infty} Tx_{2n-1} = \lim_{n \rightarrow \infty} y_{2n} = z \tag{2.8}$$

and

$$\lim_{n \rightarrow \infty} gx_{2n-1} = \lim_{n \rightarrow \infty} Sx_{2n-2} = \lim_{n \rightarrow \infty} y_{2n-1} = z. \tag{2.9}$$

First assume that the pair (S, f) is *rc* and compatible, while (T, g) is weakly compatible. Then (2.8) and (2.9) give

$$\lim_{n \rightarrow \infty} Sfx_{2n} = Sz \quad \text{and} \quad \lim_{n \rightarrow \infty} fSx_{2n} = fz.$$

Since (S, f) is compatible, it follows that $fz = Sz$.

The coincidence point z of f and S will be a common fixed point for them. In fact, writing $x = z$ and $y = x_{2n-1}$ in (2.2)

$$\begin{aligned}
 & [1 + \beta d(fz, gx_{2n-1})]d(Sz, Tx_{2n-1}) \\
 & \leq \alpha \max\{\beta d(fz, gx_{2n-1}), d(fz, Sz), d(gx_{2n-1}, Tx_{2n-1}), \\
 & \quad \frac{1}{2}[d(fz, Tx_{2n-1}) + d(Sz, gx_{2n-1})] \\
 & \quad + \beta[d(fz, Sz)d(gx_{2n-1}, Tx_{2n-1}) + d(fz, Tx_{2n-1})d(Sz, gx_{2n-1})]\}.
 \end{aligned}$$

Using (2.8) and (2.9) and $fz = Sz$, this gives

$$\begin{aligned}
 & [1 + \beta d(fz, z)]d(Sz, z) \\
 & \leq \alpha \max\{\beta d(fz, z), d(fz, Sz), 0, \frac{1}{2}[d(fz, z) + d(Sz, z)] \\
 & \quad + \beta[d(fz, Sz).0 + d(fz, z)d(Sz, z)]\}.
 \end{aligned}$$

Cancelling the common term $\beta d(fz, z)d(Sz, z)$ both sides, we get $d(Sz, z) \leq \alpha d(Sz, z)$ or $Sz = z$. Thus $fz = Sz = z$.

In view of the inclusion $S(X) \subset g(X)$, we can find a point $u \in X$ such that $Sz = gu$. Thus

$$fz = Sz = gu = z. \tag{2.10}$$

Then taking $x = z$ and $y = u$ in (2.2) and using (2.10), we have

$$\begin{aligned}
 & [1 + \beta d(fz, gu)]d(Sz, Tu) \\
 & \leq \alpha \max\{\beta d(fz, gu), d(fz, Sz), d(gu, Tu), \\
 & \quad \frac{1}{2}[d(fz, Tu) + d(Sz, gu)] \\
 & \quad + \beta[d(fz, Sz)d(gu, Tu) + d(fz, Tu)d(Sz, gu)]\}.
 \end{aligned}$$

or $d(z, Tu) \leq \alpha d(z, Tu)$ or $z = Tu$.

This together with (2.10) gives

$$fz = Sz = gu = Tu = z. \tag{2.11}$$

Since weakly compatible maps T and g commute at the coincidence point u , it follows that $gTu = Tgu$, that is $gz = Tz$.

Finally (2.2) with $x = y = z$ and (2.11) gives

$$\begin{aligned}
 & [1 + \beta d(fz, gz)]d(Sz, Tz) \\
 & \leq \alpha \max\{\beta d(fz, gz), d(fz, Sz), d(gz, Tz), \\
 & \quad \frac{1}{2}[d(fz, Tz) + d(Sz, gz)] \\
 & \quad + \beta[d(fz, Sz)d(gz, Tz) + d(fz, Tz)d(Sz, gz)]\}
 \end{aligned}$$

or $d(z, gz) \leq \alpha d(z, gz)$ so that $gz = z$.

This proves that z is a common fixed point of f, g, S and T .

The same conclusion can also be obtained when (T, g) is *rc* and compatible.

The uniqueness of the common fixed point follows directly from (2.2). □

Taking $\beta = 0$ in (2.2), we get

Corollary 1. Let f, g, S and T be self-maps on a complete metric space X satisfying the inclusions (2.1) and the inequality

$$\begin{aligned}
 & d(Sx, Ty) \leq \alpha \max\{\beta d(fx, gy), d(fx, Sx), d(gy, Ty), \\
 & \quad \frac{1}{2}[d(fx, Ty) + d(Sx, gy)]\} \\
 & \text{for all } x, y \in X, \tag{2.12}
 \end{aligned}$$

where $0 \leq \alpha < 1$. Suppose that either of the pairs (S, f) and (T, g) is *rc* and compatible, while the other weakly compatible. Then all the four maps f, g, S and T will have a unique common fixed point.

Remark 1. Singh and Mishra [6] in their result considered self-maps f , g , S and T in the complete metric space X satisfying the inclusions (2.1) and the inequality (2.12) but that (S, f) is rc and compatible, while (T, g) weakly compatible, and hence the common fixed point of them follows immediately from Corollary 1.

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