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Post Buckling Behaviour of a Nanobeam considering both the surface and nonlocal effects*

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ABSTRACT

Nano-scale beams and plates have been the key components of the sensor and actuator in nanoelectromechnical (NEMS) systems with wide applications in environmental monitoring, medical diagnostics, food processing, mining, bioengineering and defence. Nonlocal and surface effects have been incorporated to find critical load of a nano beam subjected to a transverse loading. The Nonlocal theory, expresses the stress field at a point in an elastic continuum in terms of not only strains at that point but also the strains throughout the body. The governing equation of a normal beam has been modified to achieve the governing differential equation of a nano beam. The post buckling behaviour of a nano beam has been tried to be assessed. The results showed that the surface effects try to delay the buckling process whereas the nonlocal effects contribute to the instability.

Keywords : Nano beam, Post Buckling behavior, Non local effect, Surface effect.

1 INTRODUCTION

Nano-scale beams and plates have been the key components of the sensor and actuator in nanoelectromechnical (NEMS) systems [1]. It has wide applications in environmental monitoring, medical diagnostics, food processing, mining, bioengineering and defence [2-6]. Unlike MEMS (Micro-Electro Mechanical System) technology, the appearance of NEMS is quite recent. Understanding the mechanical behaviour of such structures is thus important for the betterment of their applications in designing such systems. Varieties of materials such as Si, Al, C, SiC, are among commonly used material for fabrication of such structures.

A particularly acclaimed property of nano-scale structures is its size-dependent properties and high surface area to bulk volume ratio [7-10]. Thus length-scale effects play a key role in deciding the mechanical behavior of such system. Moreover, the energy associated with surface becomes of comparable magnitude to the bulk. The excess energy associated with the surface atoms is referred as surface free energy. The surface energy is linked to the surface-stresses analogues to the surface tension force in fluid with the assumption that the surface has thickness of several atomic layers to introduce the notion of surface stress and strains.

The size-effects have been incorporated in the classical continuum theory by rewriting the stress-strain behavior including the length scale effects, referred as nonlocal theory. The surface stress has also been linked to such calculations.

Static and dynamic behavior of Nano scale structures have been researched by investigators, but the stability aspect (both static and dynamic) has not been considered in the past studies. Liu, Rajapakse and Phani [11] presented the critical buckling load under surface stress only. A close form expression is provided for the axial buckling load for nonuniform nanowires using non-local elasticity theory in conjunction with the surface effects by Lee and Chang [12]. The effect of surface stress on post buckling behavior of nanowires has been recently discussed analytically by Li et al. [13].

In this study the post buckling behaviour of a Nano-beam, incorporating both the non-local and surface effects has been studied.

2 FORMULATION

2.1 Governing equation of Non local beam

The Nonlocal theory, as proposed by Eringen [14, 15] expresses the stress field at a point in an elastic continuum in terms of not only strains at that point but also the strains throughout the body, hence referred as 'nonlocal' theory. The local stress tensor $[t]$ at point $\{x\}$ in a linear elastic solid can be related to strain $[\varepsilon]$ at any point by Generalised Hooke's law as

$$[t]=[C]:[\varepsilon] \tag{1}$$

where, $[C]$ is the fourth order elasticity tensor and : is the contraction operator.

Following the nonlocal model, the stress tensor $[\sigma]$ at point $\{x\}$ is expressed as

$$[\sigma]=\int_V \lfloor K(|x'-x|),\tau \rfloor [t(x')] dx' \tag{2}$$

where, $[t]$ being the local stress tensor at point $\{x\}$ and the kernel of the integral is the nonlocal modulus, $|x-x'|$ is the distance and τ is the characteristic length representative of lattice spacing. The integral form of the constitutive relation makes the formulation of nonelasticity problem cumbersome. An equivalent differential form is expressed as

$$\left(1-\tau^2 l^2 \nabla^2\right)\sigma = t \quad (3)$$

where, $\tau = e_0 a/l$: e_0 is a material constant, a and l are the internal and external characteristic lengths respectively.
Considering a beam with X axis along the longitudinal direction and the depth along Z axis, the governing equation for the axially loaded beam is written as

$$\frac{\partial V}{\partial x} + q = m\frac{\partial^2 W}{\partial t^2} \quad (4)$$

$$\frac{\partial M}{\partial x} - P\frac{\partial w}{\partial x} = V \quad (5)$$

where, q is the intensity of loading, V is the shear force, M is the bending moment and P is the axial compressive force. Combining equation (4) and (5), it is obtained as

$$\frac{\partial^2 M}{\partial x^2} - P\frac{\partial^2 w}{\partial x^2} + q = m\frac{\partial^2 w}{\partial t^2} \quad (6)$$

where, M is the bending moment in the beam, V is the shear force, P is the axial force and w is the vertical deflection of the beam. Following nonlocal theory, the stress-strain relation becomes

$$\sigma_{xx} - \mu\frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\varepsilon_{xx} \quad (7)$$

Multiplying equation (6) with z^2 and integrating over the depth, the moment curvature relation is obtained as

$$EI\frac{\partial^2 W}{\partial x^2} = -\left(1-\mu\frac{\partial^2}{\partial x^2}\right)M \quad (8)$$

Differentiating equation (8) twice w.r.t x and substituting equation (6) into (8) the following governing beam bending equation is found:

$$EI\frac{\partial^4 w}{\partial x^4} - \left(1-\mu\frac{\partial^2}{\partial x^2}\right)\left\{q - P\frac{\partial^2 w}{\partial x^2} - m\frac{\partial^2 w}{\partial t^2}\right\} = 0 \quad (9)$$

2.2 Effects of surface stress

The effects of surface stresses on the beam are assumed to be governed by the Gurtin Murdoch theory of surface elasticity [16, 17]. The model while applied for a beam assumed that the beam is consisting of surfaces which are bonded to bulk .

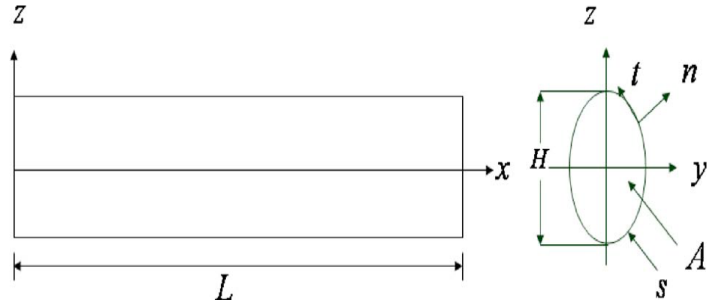

Figure 1: A prismatic beam with length L and height H set in Cartesian coordinates

While the state of stress in the bulk and associated constitutive behavior has already been described by Nonlocal theory (equation 7 and 8), the surface elastic behavior is associated with surface stress components $\left(\sigma^s_{xx}, \sigma^s_{zx}\right)$. The constitutive relations are given by

$$\sigma^s_{xx} = \sigma_0 + \left(2\mu_0 + \lambda_0\right)u_{x,x}$$

$$\sigma^s_{zx} = \sigma_0 u_{z,x}$$

(10) **[Liu et. al]**

In which, σ_0 is the residual surface stress under unrestrained conditions acting along x direction at the surface, μ_0, λ_0 are the surface Lame constants and u_x and u_z are the displacements along the X and Z direction respectively.

The stress component σ_0 will contribute in the axial force in the beam whereas, the surface shear stress will be acting perpendicular to the bulk and surface (i.e. along direction of z). With the surface stress being included, the governing equation in (9) be modified as

$$\left(EI + \left(2\mu_0 + \lambda_0\right)I^* - \frac{\nu I\sigma_0}{h}\right)\frac{\partial^4 w}{\partial x^4} - \left(1-\mu\frac{\partial^2}{\partial x^2}\right)\times \left\{q - \left(P - \sigma_0 s^*\right)\frac{\partial^2 w}{\partial x^2} - \left(m + \rho_0 s^*\right)\frac{\partial^2 w}{\partial t^2}\right\}$$

(11.a) **[Liu et. al]**

We define

$$E\bar{I} = EI + \left(2\mu_0 + \lambda_0\right)I^* - \frac{\nu I\sigma_0}{h} \quad (11.b)$$

$$\bar{P} = P - \sigma_0 s^* \quad (11.c)$$

where, $$I^* = \int_s z^2 ds \quad (12)$$

referred as the perimeter moment of inertia, s is measured along the perimeter of the section.

For rectanglular and circular cross section:

$$I = 2bh^3/3 \qquad I^* = 2bh^2 + 4h^3/3$$
$$I = \pi D^4/64 \qquad I^* = \pi D^3/8 \tag{13}$$

s^* is given by

$$s^* = \int_s n_z^2 ds \tag{14}$$

where, n_z is the direction cosine of the normal to the surface s with respect to axis z.

$$s^* = 2b \qquad H = 2h$$
$$s^* = \pi D/2 \qquad H = D \tag{15}$$

2.3 Solution of Differential Equation

In this study dynamic analysis has not been done. So we have no contribution of the last term in equation (11).

This study focuses on solving the differential equation presented in (11) by assuming a suitable function for 'w' i.e displacement function. The beam is assumed to be fixed fixed as shown in

Figure 2.

$$w = A/2(1+\cos(2\pi x/L)) \tag{16}$$

$$q = q_{transverse} + 2\sigma o D \frac{\partial^2 w}{\partial x^2} \tag{17}$$

Now substituting w from equation (16) into equation (11.b) we obtain the amplitude of the displacement (A) as eqn (18), where the denominator approaches zero for a value of P = Pcr which gives eqn(19). In the absence of any external load the amplitude of the displacement becomes Ao (eqn 20

$$A = 2q/(K\text{^}2 * \cos(Kx)(E\bar{I} + 2\mu\sigma_0 D + \sigma_0 s * -P\mu)K\text{^}2 + (P - \sigma_0 s * -2\tau D)) \tag{18}$$

where, $K = 2\pi/L$

The denominator approaches zero for a value of P = Pcr, which gives

$$Pcr = ((E\bar{I} + 2\mu\sigma_0 D)K\text{^}2 - 2\sigma o D)/(\mu K\text{^}2 - 1) + \sigma o s* \tag{19}$$

In the absence of any axial load P, the amplitude of displacement is found to be Ao

$$Ao = 2q/(K^2 \times \cos(Kx)(E\bar{I} + 2\mu\sigma_0 D + \sigma_0 s*)K\text{^}2 + (-\sigma_0 s * -2\sigma_0 D)) \tag{20}$$

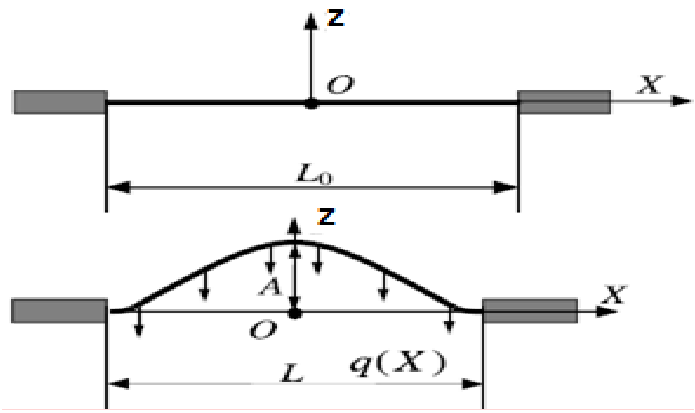

Figure 2: Schematic Diagram of a nanobeam buckling showing undeformed and buckled shape

The buckling load expressions were also found considering only surface effects, only nonlocal effects and neglecting both effects.

A, Pcr and Ao considering different beam behaviour could be found by the following:

For only surface effects, put μ = 0 in eqn (18),(19) and (20).

For only Non local effects, put σ = s* = 0 and $\bar{I} = I = \pi D^4/64$ in eqn(18), (19) and (20).

Neglecting both effects we get the expression of Critical Buckling load same as Critical Buckling Load for a fixed fixed beam i.e, $Pcr = EIK\text{^}2 = 4\pi\text{^}2EI/L\text{^}2$

TABLE 1
BUCKLING LOAD FORMULAE

Beam Behaviour	**Buckling load (Pcr)**
Nonlocal and surface effects considered	$(EI + 2\mu\sigma_0 D)(K^2 - 2\sigma_0 D)/(\mu K^2 - 1) + \sigma_0 s^*$
Only Surface effects considered	$(EIK\text{^}2 + 2\sigma_0 D) + \sigma_0 s*$
Only Non local effects considered	$(EIK^2)/(\mu K^2 - 1)$
Normal Beam [neglecting both surface and nonlocal effects]	$EIK^2 = 4\pi^2 EI/L^2$

3 Results and Discussions

The basic idea of this study is to study the post buckling behaviour of a Nano-beam under the presence of transverse loading q (17). Taking an aluminium Nano-beam as an example to show the post buckling behaviour, the modelling parameters are used as given in Table 1, Liu et.al[22]. Length = 200e-9(m), D = 20e-9(m).

There was also transverse load acting of magnitude 20N/m. As seen from the expressions in Table 1, the transverse load has no contribution in the Buckling load. The transverse load only influences the amplitude of deformation as shown in (18) and (20).

TABLE 2
Material Properties

Material Properties	Al	Si
E(GPa)	90	107
ν	0.23	0.33
μ_0(N/m)	-5.4251	-2.7779
λ_0(N/m)	3.4939	-4.4939
Γ_0(N/m)	0.5689	0.6056
ρ(kg/m^3)	2.7x10^3	2.33x10^3
ρ_0(kg/m^3)	5.46x10^{-7}	3.17x10^{-7}

I, I*, s* are calculated from equations (13) and (15) for circular cross section. Since dynamic analysis has not been done, the ρ and ρ_o are not of use to us.

TABLE 3
Buckling loads for different cases

Beam Behaviour	Buckling load (Pcr)
Nonlocal and surface effects considered	4.062852060627220e-08
Only Surface effects considered	6.343737091346891e-07
Only Non local effects considered	1.260659248282454e-22

These values support the fact that surface effects tend to act like restoring forces, and nonlocal effect tends to help in the buckling phenomenon. As a result when only surface effects are considered, the buckling load value comes larger than the value obtained when both effects are considered. Also when surface effects are absent, the Buckling load value comes out to be very less

Curve between A/Ao (normalized displacement) vs P/Pcr (normalized load) was plotted in Figure 3 and Figure 4.

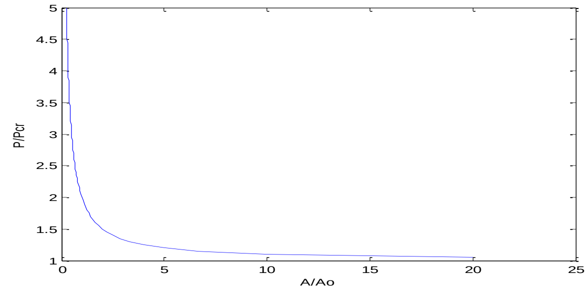

Figure 3: Load Displacement curve for P > Pcr

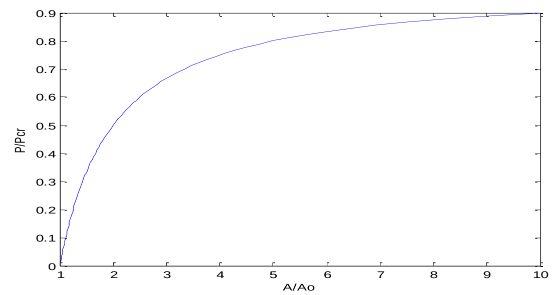

Figure 4: Load Displacement curve for P < Pcr

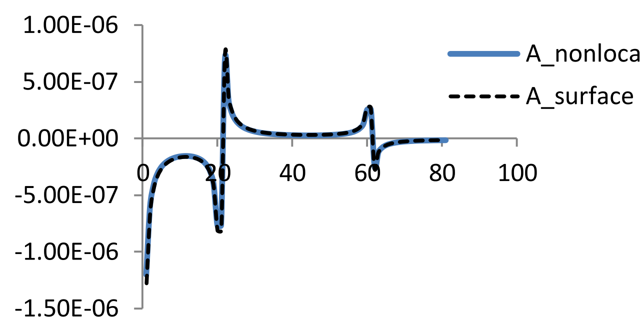

Figure 5: Plot of A_nonlocal vs A_surface (plotted on the same axis)

A plot has been obtained by increasing the load (P) linearly and finding the corresponding values of A, considering only nonlocal behaviour (A_nonlocal) and considering only surface behaviour (A_surface).as shown in Figure 5. We find that both the curves have overlapped each other completely. This means that the values of A coming from only nonlocal behaviour and only surface effect match each other.

To see the effect of decreasing the cross section size even more, a graph was plotted between P (function of L/D, where L=length of beam and D=diameter of c/s) normalized by Pcr (considering both non local and surface effect) vs L/D. The curve obtained is shown in Figure 6.

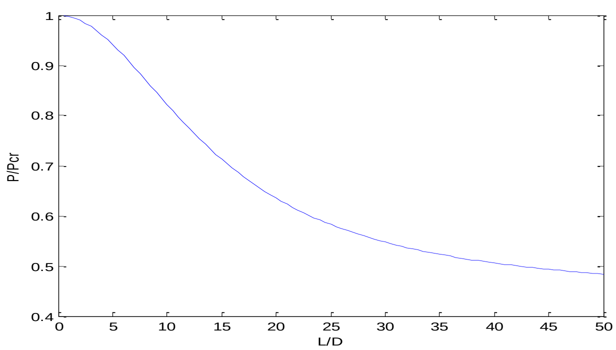

Figure 6: Variation Of Buckling load with L/D ratio.

We find that as the L/D ratio increases the P/Pcr value comes down as shown in Figure 6 It implies, for a common value of D, as the length of the nano beam is increased, the buckling load value decreases, i.e the buckling occurs before, which matches with our understanding (as the section becomes slender the buckling load value decreases.

4 Conclusions

- The surface effects in a nano beam act as stabilising forces. It delays the buckling phenomenon for a beam. The Critical load values as tabulated in Table 3 show that when surface effects are only present, the buckling load value is the maximum. Also for only non-local effects, the buckling load value is very very small. Thus, the non local effects act as destabilising forces in a nano beam.
- It is more like surface tension effect which too is a restoring force.
- The post buckling behaviour is shown in Figure 3. Where we see that after buckling the section again gets restored and the value of max amplitude decreases with increase in load. This behaviour has to be verified and was beyond the scope of my work.

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References

[1] Wang X., Ozkan C. S.: "Multisegment nanowire sensors for the detection of DNA molecules", Nano Letters., 2008, 8, 398-404.
[2] Stewart M. E., Anderton C. R., Thompson L. B., Maria J., Gray S. K, Rogers J A and Nuzzo R. G. Chemical Review, 2008, 108, 494-521.
[3] Venkatesan B. M., Dorvel B., Yemenicioglu S, Watkins N., Petrov I., and Bashir R., 2009 Advance Materials. 21, 2771-6.
[4] Ahn M. W., Park K. S. Heo J. H., Park J. G., Kim D. W., Choi K. J., Lee J. H. and Hong S. H. 2008 Applied Phys letter, 93,263103
[5] Kauffman D R and Star A 2008, Agnew Chem. Int. Edn 47,6550-70.
[6] Lee J-H 2009 Sensors and Actuators B 140, 319-36.
[7] Miller R. E. and Shenoy V. B. 2000, Size-dependent elastic properties of nanosized structural elements Nanotechnology,11,139.
[8] Shenoy V. B. Size-dependent rigidities of nanosized torsional elements, 2002
[9] Chen C. Q., Shi Y., Zhang Y. S., Zhu J. and Yan Y. J. [2006] Size dependence of Young's Modulus in ZnO nanowires, Physics Letters, 96, 075505.
[10] Miller R. E. and Shenoy V. B. [2000] Size-Dependent Elastic Properties of Nanosized Structural Elements, Nanotechnology, 11,3,139-147.
[11] Liu C., Rajapakse R. K. N. D., Phani A. S. [2011] Finite Element Modelling of Beams With Surface Energy Effects, Journal of Applied Mechanics, ASME, 78, 1-9.
[12] Lee H. L., Chang W. J. [2010] Surface effects on axial buckling of nonuniform nanowires using nonlocal elasticity theory, Micro and Nano Letters, 6,1, 19-21.
[13]Li Yuhang, Song J., Fang B., Zhang J. [2011] Surface effects on the postbuckling of nanowires, 44, 425304.
[14] Eringen A. C. [1972] Nonlocal polar elastic continua, International Journal of Engineering Science, 10, 1-16.
[15] Eringen A. C., Edelen D. G. B., [1972] On nonlocal Elasticity, International Journal of Engineering Science, 10,233-248.
[16] Gurtin M. E. and Murdoch A. I. [1975] A continuum Theory of Elastic Material Surfaces, Archive of Rational Mechanics Analysis, 57, 291-323.
[17] Gurtin M. E. and Murdoch A. I. [1975] Addenda to Our Paper: A Continuum Theory of Elastic Material Surfaces, Archives of Rational Mechanics Annals, 59, 389-390.

