Optimal Equi-scaled Families of Ostrowski's Method System of Nonlinear Equations

Sandeep Singh

Mathematics, D. A. V. College, Chandigarh, India.
Email:sbisht402357@gmail.com

ABSTRACT

In this paper, we present many new fourth -order optimal families of Ostrowski's method for computing zeros of system of nonlinear equations numerically. In this paper, we extending the idea of the proposed families of Ostrowski's method to system of nonlinear equations .It is proved that the above said families have fourth order of convergence. Several numerical examples are also given to illustrate the efficiency and the performance of the presented families.

Keywords: System of Nonlinear equations, Optimal Order of Convergence, Halley's method, Schroder's method, Ostrowiski's method.

1 INTRODUCTION

Due to the fact that systems of nonlinear equations arise frequently in science and engineering they have attracted researcher's interest. For example, nonlinear systems of equations, after the necessary processing step of implicit discretization, are solved by finding the solutions of systems of equations. We consider here the problem of finding a real zero, $x^* = (x^*_1, x^*_2 \ldots\ldots; x^*_n)^T$, of a system of non linear equations

$$f_1(x_1, x_2, \ldots\ldots\ldots, x_n) = 0;$$
$$f_2(x_1, x_2, \ldots\ldots\ldots, x_n) = 0;$$
$$\vdots$$
$$\vdots$$
$$f_n(x_1, x_2, \ldots\ldots\ldots, x_n) = 0;$$

This system can referred in vector form by

$$F(X) = 0 \qquad (1.1)$$

Where $F = (f_1, f_2, \ldots\ldots\ldots, f_n)^T$ and
$X = (x_1, x_2, \ldots\ldots\ldots, x_n)^T$

Let the mapping $F: D \subseteq R^n \rightarrow R^n$ assumed to satisfy the assumptions (1.1). F(X) is continuously differentiable in an open neighborhood D of X*. There exists a solution vector X* of (1.1) in D such that F(X*) = 0 and F'(X*) ≠ 0. Then the standard method for finding the solution to equation (1.1) is the classical Newton's method [2-5] given by

$$X^{k+1} = X^k - F(X^k)/F'(X^k), \quad k = 0, 1, 2, \ldots\ldots\ldots \quad (1.2)$$

2 CONSTRUCTION OF NOVEL TECHNIQES WITHOUT MEMORY

Case I: New optimal families of Ostrowski's method. The well-known Schroder's method [9] for multiple zero and Halley's method [6] for simple zero, are given by

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f'^2(x_n) - f(x_n)f''(x_n)} \qquad (2.1)$$

and

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)} \qquad (2.2)$$

respectively.

We now intend to develop new optimal families of Ostrowski's method the having the cubing scaling factor of function in the correction factor. For this, we take the Newton's method

$$y_n = x_n - f(x_n)/f'(x_n) \qquad (2.3)$$

Similarly using the concept, we now expand the function $f(y_n) \cong f(x_n - f(x_n)/f'(x_n))$ about the point $x = x_n$ by Taylor's series expansion, we have

$$f(x_n - f(x_n)/f'(x_n)) \cong \frac{f^2(x_n)f''(x_n)}{2f'^2(x_n)}$$

therefore, we obtain

$$f''(x_n)) \cong \frac{f^2(x_n)f''(y_n)}{2f'^2(x_n)} \qquad (2.4)$$

$$x_{n+1} = x_n - \frac{1}{2}\left[\frac{f(x_n)f'(x_n)}{f'^2(x_n) - f(x_n)f''(x_n)} + \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}\right] \qquad (2.5)$$

Using the value of $f''(x_n)$ from(2.4) in the equation(2.5), we get

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ x_{n+1} = x_n - \dfrac{1f(x_n)}{2f'(x_n)}\left[\dfrac{f(x_n)}{f(x_n)-2f(y_n)} + \dfrac{f(x_n)}{f(x_n)-f(y_n)}\right] \end{cases} \quad (2.6)$$

This method has quadratic convergence and satisfies the following error equation

$$e_{n+1} = -(c_2^2/2)\, e_n^2 + ((3/2)c_2^2 - c_3)\, e_n^3 + O(e_n^4)$$

Again according to the Kung-Traub conjecture [7], the above method (2.6) is not an optimal method because it has second-order convergence and requires three evaluations of function per full iteration. Therefore, to built our optimal families of Ostrowski's method, we take five free disposable parameters. Therefore, we consider

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ x_{n+1} = x_n - \dfrac{1f(x_n)}{2f'(x_n)}\left[\dfrac{b_1 f(x_n)}{b_2 f(x_n)-2f(y_n)} + \dfrac{b_3 f(x_n)}{b_4 f(x_n)-b_5 f(y_n)}\right] \end{cases} \quad (2.7)$$

Where b_1, b_2, b_3, b_4, b_5 are disposable parameters such that the order of convergence reaches at the optimal level four without using any more functional evaluations. [10] Indicates that under what choice on the disposable parameters in (2.6), the order of convergence will reach at the optimal level four.

Where

$$b_1 = 4(b_4-b_5)^3/(4b_4^3 - 6b_4^2 b_5 + 4b_4 b_5^2 - b_5^3),$$
$$b_2 = 2(b_4-b_5)/(2b_4-b_5),\ b_5 \neq 2b_4, \quad (2.8)$$
$$b_3 = 2b_4^3/2b_4^2 - 2b_4 b_5 + b_5^2,$$

Where $b_4, b_5 \epsilon R$ but b_4 and b_5 such that neither $b_4 = 0$ nor $b_4 = b_5$. Put all these values in (2.6), we have

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ x_{n+1} = x_n \\ \quad - \dfrac{1f(x_n)}{2f'(x_n)}\left[\dfrac{(b_4^2 + b_{4}b_5 - b_5^2) f(x_n) f(y_n) - b_4(b_4-b_5) f^2(x_n)}{(b_4 f(x_n) - b_5 f(y_n))((2b_4-b_5) f(y_n) - (b_4-b_5) f(x_n))}\right] \end{cases} \quad (2.9)$$

Now extend this idea for system of equations, we have

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ x_{n+1} = x_n \\ \quad - \dfrac{F(X_n)}{2F'(X_n)}\left[\dfrac{[(b_4^2 + b_{4}b_5 - b_5^2) F(Y_n) - b_4(b_4-b_5) F(X_n)]^T}{(b_4 F(X_n) - b_5 F(Y_n))^T((2b_4-b_5) F(Y_n) - (b_4-b_5) F(X_n))}\right] F(X_n) \end{cases} \quad (2.10)$$

3 COVNERGENCE ANALYSES

We shall present the mathematical proof for the order of convergence of formula (2.10).

Lemma 1: *Let $D \subseteq R^n \to R^n$ be P-times Frechet differentiable in a convex set $D \subseteq R^n$ then for any X, $H \epsilon R^n$, the following expression holds:*

$$F(X+H) = F(X) + F'(X)H + 1/2!\ F''(X)H^2 + 1/3!\ F'''(X)H^3 + \ldots + 1/(p-1)!F^{p-1}(X)H^{p-1} + R_p. \quad (3.1)$$

Where

$$||R_p|| \leq 1/p!\ \sup ||Fp(X+tH)||\,||H||^p,\quad 0 \leq t \leq 1$$

and $Hp = (h, h, \ldots\ldots^p, \ldots, h)$.

Now we analyze the behavior of (2.10) through the following theorem:

Theorem 3.1. *Let $D \subseteq R^n \to R^n$ be four times Frecht differentiable in a convex set D containing the root r of F(x) = 0. Then, the sequence x_k, $k \leq 0$ ($x^0 \epsilon D$) obtained by using the iterative expression of method (2.10) converges to r with convergence order four if $b_4 \neq 0$ and $b_4 \neq b_5$.*

Proof: The Taylor's expansion (3.1) for F(x) about x^k is

$$F(x) = F(x^k) + F'(x^k)(x-x^k) + 1/2!\ F''(x^k)(x-x^k)^2 + 1/3!\ F'''(x^k)(x-x^k)^3 + 1/4!\ F^{iv}(x^k)(x-x^k)^4 + O(||x-x^k||^5). \quad (3.2)$$

Let $e^k = x^k - r$. Then, setting x = r and using F(r) = 0 in (3.2), we obtain

$$F(x^k) = F'(x^k)e^k - 1/2!\ F''(x^k)(e^k)^2 + 1/3!\ F'''(x^k)(e^k)^3 - 1/4!\ F^{iv}(x^k)(e^k)^4 + O(||e^k||^5). \quad (3.3)$$

Pre-multiplying by $F'(x^k)^{-1}$ to both sides of (3.3)

$$F'(x^k)^{-1}F(x^k) = e^k - 1/2!\ F'(x^k)^{-1}F''(x^k)(e^k)^2 + 1/3!\ F'(x^k)^{-1}F'''(x^k)(e^k)^3 - 1/4!\ F'(xk)^{-1}F^{iv}(x^k)(e^k)^4 + O(||e^k||^5). \quad (3.4)$$

Now from (2.10), we yields

$$y^k - x^k = -ek + 1/2!\ F'(x^k)^{-1}F''(x^k)(e^k)^2 - 1/6\ F'(x^k)^{-1}F'''(x^k)(e^k)^3 + (||(e^k)||)^4. \quad (3.5)$$

Also,

$$(y^k - x^k)^2 = (e^k)^2 - F'(x^k)^{-1}F''(x^k)(e^k)^3 + (||(ek)||)^4, \quad (3.6)$$
$$(y^k - x^k)^3 = -(e^k)^3 + (||(e^k)||)^4, \quad (3.7)$$
$$(y^k - x^k)^4 = (||(e^k)||)^4. \quad (3.8)$$

Here $e^m = (e, e, \ldots\ldots^m, \ldots, e)$, $e \epsilon R^n$.

Taylor's expansion of $F(y^k)$ about x^k is.

$$F(y^k) = F(x^k) + F'(x^k)(y^k-x^k) + 1/2!F''(x^k)(y^k-x^k)^2 + 1/3!\ F'''(x^k)(y^k-x^k)^3 + O(||y^k-x^k||^4). \quad (3.9)$$

From (3.5)-(3.8) and (3.9), we obtain

$$F(y^k) = F(x^k)-F'(x^k)\, e^k + F''(x^k)(e^k)^2-[1/3F'''(x^k) +1/2F''(x^k)F'(x^k)^{-1}F''(x^k)(e^k)^3]+O(||e^k||^4). \quad (3.10)$$

Now from(2.10)

$$e^{k+1} = e^k - F'(x^k)^{-1} F(x^k)\left[\frac{[(b_4{}^2+b_4b_5-b_5{}^2)F(\mathrm{Y^k})-b_4(b_4-b_5)F(\mathrm{X^k})]^T}{(b_4F(\mathrm{X^k})-b_5F(\mathrm{Y^k}))^T((2b_4-b_5)F(\mathrm{Y^k})-(b_4-b_5)F(\mathrm{X^k})}\right]F(x^k)$$

Let $R(x^k) = F'(x^k)^{-1} F(x^k))$ $[(b_4{}^2 + b_4b_5 - b_5{}^2)F(\mathrm{Y^k}) - b_4(b_4 - b_5)F(\mathrm{X^k})]^T F(\mathrm{X^k})$

and $w(x^k) = (b_4F(\mathrm{X^k}) - b_5F(\mathrm{Y^k}))^T((2b_4 - b_5)F(\mathrm{Y^k}) - (b_4-b_5)F(\mathrm{X^k}))$

$$R(x^k) = (b_4(b_5 - b_4)(F'(x^k))^TF'(x^k)(e^k)^3 + [(3/2b_5{}^2- 2b_4b_5 -b_4{}^2)(F''(x^k))^TF'(x^k) +1/2\,(b4^2 - b_4b_5)(F'(x^k))^TF'(x^k) +1/2b_4(b_4 - b_5)\, F'(x^k)^{-1}F''(x^k)(F'(x^k))^TF'(x^k)](e^k)^4 +O(||e^k||^5)$$

$$w(x^k) = b_4(b_5 - b_4)(F'(x^k))^TF'(x^k)(e^k)^2 + [1/2\,(3b_4 - 2b_5)b_4(F'(x^k))^TF''(x^k) -(b_4 + b_5)(b_4 - b_5)(F''(x^k))^TF'(x^k)](e^k)^3 + O(||e^k||^4). \quad (3.12)$$

From (3.11) and (3.12), we obtain

$$w(x^k)e^k - R(x^k) = [1/2\,(5b_4{}^2 - 3b_4b_5)(F'(x^k))^TF''(x^k) +(2b_4{}^2- (5/2)b_5{}^2 +2b_4b_5)(F''(x^k))^TF'(x^k) - 1/2(b_4{}^2 - b_4b_5)F'(x^k)^{-1}F''(x^k)(F'(x^k))^TF'(x^k)](e^k)^4 +O(||e^k||^5). \quad (3.13)$$

From(2.10) and (3.13), we have

$$w(x^k)\, e^{k+1} = w(x^k)\, e^k - R(x^k) = [1/2\,(5b_4{}^2 - 3b_4b_5)(F'(x^k))^TF''(x^k) +(2\, b_4{}^2 -5/2b_5{}^2 +2b_4b_5)\, (F''(x^k))^TF'(x^k) -1/2(b_4{}^2 - b_4b_5)F'(x^k)^{-1}F''(x^k)(F'(x^k))^TF'(x^k)]\, (e^k)^4 + O(||e^k||^5). \quad (3.14)$$

From the above equation it is clear that (2.10) is fourth order convergence if $b_4 \neq 0$ and $b_4 \neq b_5$.

Special cases of formula (2.10):

(a) For $b_4 = 1$ and $b_5 = 1/10$, family (2.10) read as:

$$\begin{cases} y_n = x_n - \frac{f(\mathrm{x_n})}{f'(\mathrm{x_n})} \\ x_{n+1} = x_n \\ -\frac{F(\mathrm{X_n})}{F'(\mathrm{X_n})}\left[\frac{[90F(\mathrm{X_n})-109F(\mathrm{Y_n})]^T}{(10F(\mathrm{X_n})-F(\mathrm{Y_n}))^T(9F(\mathrm{X_n})-19F(\mathrm{Y_n})}\right]F(\mathrm{X_n}) \end{cases} \quad (3.15)$$

This is a new fourth-order method and satisfies the following error equation

$$w(x^k)\, e^{k+1} = [(47/20)\, (F'(x^k))^TF''(x^k) +(87/40)\, (F''(x^k))^T F'(x^k) -(9/20)F'(x^k)^{-1}F''(x^k)(F'(x^k))^TF'(x^k)]\, (e^k)^4 + O(||e^k||^5).$$

(b) For b4 = 10 and b5 = 1 10 , family (2.10) read as:

$$\begin{cases} y_n = x_n - \frac{f(\mathrm{x_n})}{f'(\mathrm{x_n})} \\ x_{n+1} = x_n \\ -\frac{F(\mathrm{X_n})}{F'(\mathrm{X_n})}\left[\frac{[9900F(\mathrm{X_n})-10099F(\mathrm{Y_n})]^T}{(100F(\mathrm{X_n})-F(\mathrm{Y_n}))^T(99F(\mathrm{X_n})-199F(\mathrm{Y_n})}\right]F(\mathrm{X_n}) \end{cases} \quad (3.16)$$

This is a new fourth-order method and satisfies the following error equation

$$w(x^k)\, e^{k+1} = [(497/2)\, (F'(x^k))^TF''(x^k) +(819/40)\, F''(x^k))^T F'(x^k) -(99/2)F'(x^k)^{-1}F''(x^k)(F'(x^k))^TF'(x^k)]\, (e^k)^4 + O(||e^k||^5).$$

(c) For b4 = 49/100 and b5 = 1, family (2.10) read as:

$$\begin{cases} y_n = x_n - \frac{f(\mathrm{x_n})}{f'(\mathrm{x_n})} \\ x_{n+1} = x_n \\ -\frac{F(\mathrm{X_n})}{F'(\mathrm{X_n})}\left[\frac{[2499F(\mathrm{X_n})-2699F(\mathrm{Y_n})]^T}{(49F(\mathrm{X_n})-100F(\mathrm{Y_n}))^T(2F(\mathrm{X_n})-5F(\mathrm{Y_n})}\right]F(\mathrm{X_n}) \end{cases} \quad (3.17)$$

This is a new fourth-order method and satisfies the following error equation

$$w(x^k)\, e^{k+1} = [(539/4000)\, (F'(x^k))^TF''(x^k) +(7699/5000)\, (F''(x^k))^T F'(x^k) -(2499/20000)F'(x^k)^{-1}F''(x^k)(F'(x^k))^TF'(x^k)]\, (e^k)^4 + O(||e^k||^5).$$

4 NUMERICAL RESULTS

In this section, we shall check the performance of the present formula VS1(3:15), VS2(3:16) and VS3(3:17), the comparison is carried out with Newton's method and with HM and CM [8]. A mat lab program has been written to implement these methods. We use the following stopping criteria for computer programs:

1. $\epsilon = e^{-10}$.
2. $|F(X_n)| < \epsilon$

For every method, we analyze the number of iterations needed to converge to the required solution. The numerical results are reported in the Table 1.

We consider the following problems for a system of nonlinear equations.

Problem (a)

$$x_1{}^2 - 2\, x_1 - x_2 + 0.5 = 0$$
$$x_1{}^2 + 4\, x_2{}^2 - 4 = 0$$

Problem (b)

$$x_1{}^2 + x_2{}^2 -1 = 0$$
$$x_1{}^2 - x_2{}^2 + 0.5 = 0$$

Problem (c

$$x_1{}^2 - x_2{}^2 +3\log(x_1) = 0$$
$$2x_1{}^2 - x_1\, x_2 - 5x_1 + 1 = 0$$

Problem (d)

$$e^{x_1} + x_1\, x_2 - x_2 - 0.5 = 0$$

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$$\sin(x_1 x_2) + x_1 + x_2 - 1 = 0$$

Problem (e)

$$x_1 + 2x_2 - 3 = 0$$
$$2x_1^2 + x_2^2 - 5 = 0$$

Problem (f)

$$x_1 + e^{x_2} - \cos(x_2) = 0$$
$$3x_1 - x_2 - \sin(x_2) = 0$$

Problem (g)

$$x_1^2 + x_2^2 + x_3^2 - 9 = 0$$
$$x_1 x_2 x_3 - 1 = 0$$
$$x_1 + x_2 - x_3^2 =$$

Solution (a)

$r = (1.9006767263670658, 0.31121856541929427)^T$

Solution (b)

$r = (0.500000000000000000, 0.8660254378443865)^T$

$r = (-0.5000000000000000, -0.8660254378443865)^T$

Solution (c)

$r = (1.3192058033298924, -1.6035565551874148)^T$

Solution (d) $r = (0, 1)^T$

Solution (e)

$r = (1.4880338717125849, 0.75598306414370757)^T$

Solution (f)

$r = (0, 0)^T$

Solution (g)

$r = (2.2242448288477843, 0.28388497407293814, 1.58370776128252723)^T$

$r = (0.28388497407293814, 2.2242448288477843, 1.58370776128252723)^T$

TABLE 1: Numerical results of problems (a) to (g) using different methods.

F(X)	X	NM	HM	CM	VS1	VS2	VS3
(a)	$(3,2)^T$	10	7	6	3	3	3
	$(1.6, 0)^T$	9	7	6	3	3	3
(b)	$(.7,1.3)^T$	9	6	5	2	2	2
	$(-1,-2)^T$	9	6	5	3	3	3
(c)	$(0.91, -2)^T$	9	7	6	3	3	3
	$(1.8, -2.1)^T$	9	6	5	3	3	3
(d)	$(.7, .9)^T$	9	6	5	4	4	4
	$(-0.1, 0.2)^T$	9	6	6	3	3	3
(e)	$(.99, .1)^T$	9	7	6	4	4	4
	$(1.9, 1.4)^T$	9	7	5	4	4	4
(f)	$(1.5, 2)^T$	10	7	6	3	3	3
	$(.3, .5)^T$	9	6	5	2	2	2
(g)	$(.2, .6,1.5)^T$	9	6	5	4	4	4
	$(3, .5, 2)^T$	9	6	5	3	3	3

5 CONCLUSIONS

The presented formula (3.15),(3.16) and (3.17) is simple to understand, easy to program and has the fourth order of convergence. We contribute to the development of iteration processes and propose several families of Ostrowski's method. We now obtain a wide general class of Ostrowski's families which are without memory and have the same scaling factor of function as that Ostrowski's method. Numerical tests have been performed, which not only illustrate the method practically but also serve to check the validity of theoretical results we have derived. The performance is compared with Newton method, CM [8] and HM [8].

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