On Upper And Lower $\delta$-Precontinuous Fuzzy Multifunctions

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ABSTRACT

In this paper a new type of fuzzy multifunction termed as $\delta$-precontinuous fuzzy multifunction has been introduced and studied. Some characterizations and several properties of this multifunction are obtained. We also characterize this multifunction by newly define fuzzy neighbourhood of a fuzzy set in a fuzzy topological space. Finally, in the last section some applications of this fuzzy multifunction are obtained.

Keywords: Fuzzy upper (lower) $\delta$-precontinuous multifunctions, $\delta$-prenbd of a point, (fuzzy) $\delta$-preopen set, fuzzy upper (lower) nbd of a fuzzy set.

1 INTRODUCTION

In 1985, Papageorgiou [10] introduced fuzzy multifunction, a function from an ordinary topological space $X$ to a fuzzy topological space $Y$ and from then a group of researchers are engaged themselves for studying different types of fuzzy multifunctions. Papageorgiou defined upper and lower inverses of a fuzzy multifunction. Afterwards, it was noticed in [9] that the definition of lower inverse of Papageorgiou was not natural and so this notion was redefined in [9] via $q$-coincidence and $q$-neighbourhoods of Pu and Liu [11] and ultimately some expected results were achieved in [9] by use of the new definition. With the definition of upper inverse as given in [10] and the definition of lower inverse as defined in [9], in this paper we introduce fuzzy upper (lower) $\delta$-precontinuous multifunctions and give some characterizations of this specially via fuzzy uppr (lower) nbd.

Throughout this paper, $(X, \tau)$ or simply $X$ will stand for an ordinary topological space, while by $(Y, \tau_Y)$ or simply by $Y$ will always be denoted by a fuzzy topological space (fts, for short) in the sense of Chang [4]. The support of a fuzzy set $A$ in $Y$ will be denoted as $\text{supp} A$ [14] and is defined by $\text{supp} A = \{ y \in Y : A(y) \neq 0 \}$. A fuzzy point $[11]$ with the singleton support $y \in Y$ and the value $a (0 < a \leq 1)$ at $y$ will be denoted by $y_{\alpha}$ cl $A$ and int $A$ of a set $A$ in $X$ (respectively, a fuzzy set [14] in $Y$) respectively stand for the closure and interior of $A$ in $X$ (respectively, in $Y$). $y_{\infty}$ and $y_{1}$ are the constant fuzzy sets taking respectively the constant values 0 and 1 on $Y$. The complement of a fuzzy set $A$ in $Y$ will be denoted by $1_A \setminus A$ [14], defined by $(1_A \setminus A)(y) = 1 - A(y)$, for each $y \in Y$. For two fuzzy sets $A$ and $B$ in $Y$, we write $A \leq B$ iff $A(y) \leq B(y)$, for each $y \in Y$, while we write $A \wedge B$ to mean $A$ is quasi-coincident (q-coincident, for short) with $B$ [11] if there is some $y \in Y$ such that $A(y) + B(y) > 1$; the negation of $A \wedge B$ is written as $A \bar{\wedge} B$. A (fuzzy) set $A$ in $X$ (resp. in $Y$) is called fuzzy regular open if $A = \text{int cl } A [1]$. A fuzzy set $A$ in $Y$ is called a fuzzy neighbourhood (fuzzy nbd, for short) of a fuzzy set $B$ [11] if there exists a fuzzy open set $U$ in $Y$ such that $B \subseteq U \subseteq A$. A fuzzy set $B$ is called a quasi-regular nbd of a fuzzy point $x_a$ in $Y$ if there exists a fuzzy open set $V$ in $Y$ such that $x_a \in V \subseteq U$. A fuzzy set $A$ in an fts $Y$ is called a fuzzy $\delta$-pre-neighborhood ($\delta$-pre-nbd, for short) [12] of a point $x_a$ in $Y$ if there exists a fuzzy $\delta$-preopen set $V$ in $Y$ such that $x_a \in V \subseteq U$. A fuzzy set $A$ in an fts $Y$ is called a fuzzy $\delta$-precluster point of a fuzzy set $A$ in $Y$ if every fuzzy $\delta$-pre-neighborhood of $x_a$ is $q$-coincident with $A$. The union of all fuzzy $\delta$-pre-cluster points of $A$ is called the fuzzy $\delta$-closure of $A$ and is denoted by $\delta cl A [5]$. A subset (fuzzy set) $A$ is said to be $\delta$-preopen [12] (resp. fuzzy $\delta$-preopen [3]) in $X$ (in $Y$) if $A \subseteq \text{int } (\delta cl A)$ (resp. $A \subseteq \text{int } (\delta cl A)$). The family of all (fuzzy) $\delta$-preopen sets in $X$ (resp. in $Y$) is denoted by $\text{PO}(X)$ [12] (resp. $\text{PO}(Y)$ [3]). The $\delta$-preinterior of a subset $A$ [12] of $X$ is defined to be the union of all $\delta$-preopen sets contained in $A$ and is denoted by $\delta - \text{int } A$. The complement of a $\delta$-preopen set is called $\delta$-preclosed [12]. The intersection of all $\delta$-preclosed sets containing $A$ in $X$ is called $\delta$-preclosure of $A$ [12] and is denoted by $\delta - \text{pc} cl A$. A set $A$ is $\delta$-preopen ($\delta$-preclosed) iff $A = \delta - \text{pint } A$ (resp. $A = \delta - \text{pc} cl A$) [12]. A subset $U$ of $X$ is called a $\delta$-preneighbourhood ($\delta$-pre-nbd, for short) [12] of a point $x \in X$ if there exists a $\delta$-preopen set $V$ in $X$ such that $x \in V \subseteq U$. A fuzzy set $A$ in an fts $Y$ is called a fuzzy $\delta$-pre-neighborhood ($\delta$-pre-nbd, for short) [12] of a fuzzy point $x_a$ in $Y$ if there exists a fuzzy $\delta$-preopen set $V$ in $Y$ such that $x_a \in V \subseteq U$. A fuzzy set $A$ in an fts $Y$ is called a fuzzy $\delta$-pre-cluster point of a fuzzy set $A$ in $Y$ if every fuzzy $\delta$-pre-neighborhood of $x_a$ is $q$-coincident with $A$. The union of all fuzzy $\delta$-pre-cluster points of $A$ is called the fuzzy $\delta$-closure of $A$ and will be denoted by $\delta - \text{pc} cl A$.

2 SOME WELL KNOWN DEFINITIONS, THEOREMS AND LEMMAS

First we recall the following definitions from [10] for ready references.

Definition 2.1.
Let $(X, \tau)$ and $(Y, \tau_Y)$ be respectively an ordinary topological space and an fts. We say that $F : X \to Y$ is a fuzzy multifunction if corresponding to each $x \in X$, $F(x)$ is a unique fuzzy set in $Y$.

Henceforth by $F : X \to Y$ we shall mean a fuzzy multifunction in the above sense.

Definition 2.2. [10, 9]
For a fuzzy multifunction $F : X \to Y$, the upper inverse $F^+$ and lower inverse $F^-$ are defined as follows:
For any fuzzy set $A$ in $Y$, $F^+(A) = \{ x \in X : F(x) \subseteq A \}$ and $F^-(A) = \{ x \in X : F(x) \cap A \}$. The relationship between the upper and the lower inverses of a fuzzy multifunction is known to be as follows:

Theorem 2.3 [9]
For a fuzzy multifunction $F : X \to Y$, we have $F^-(1_Y \setminus A) = X \setminus F^+(A)$, for any fuzzy set $A$ in $Y$.

Definition 2.4. [4]
Let $A$ be a fuzzy set in an fts $Y$. A collection $\mathcal{U}$ of fuzzy sets in $Y$ is called a fuzzy cover of $A$ if $\operatorname{supp}(U(x)) : U \in \mathcal{U} = 1$, for each $x \in \operatorname{supp} A$. If, in addition, the members of $\mathcal{U}$ are fuzzy open (regular open), then $\mathcal{U}$ is called a fuzzy open (resp., regular open) cover of $A$. In particular, if $A = 1_Y$, we get the definition of fuzzy cover (open cover, regular open cover) of the fts $Y$.

Definition 2.5. [6]
A fuzzy cover $\mathcal{U}$ of a fuzzy set $A$ in an fts $Y$ is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 \supseteq A$. Clearly, if $A = 1_Y$ in particular, then the requirements on $\mathcal{U}_0$ is $\mathcal{U}_0 \supseteq 1_Y$.

Definition 2.6. [6]
An fts $Y$ is said to be fuzzy compact if every fuzzy open cover of $Y$ has a finite subcover.

Lemma 2.7. [12]
Let $A$ be a subset of a space $(X, \tau)$. Then $A \in \delta - PO(X)$ iff $A \cap U \subseteq \delta - PO(X)$ for each regular open (δ-open) set $U$ of $X$.

Lemma 2.8. [12]
Let $A$ and $X_0$ be subsets of a space $(X, \tau)$. If $A \in \delta - PO(X)$ and $X_0$ is δ-open in $(X, \tau)$, then $A \cap X_0 \in \delta - PO(X_0)$.

Lemma 2.9. [12]
Let $A \subseteq X_0 \subseteq X$. If $X_0$ is δ-open in $(X, \tau)$ and $A \in \delta - PO(X_0)$, then $A \in \delta - PO(X)$.

Definition 2.10. [7]
A topological space $(X, \tau)$ is called semi-regular if $\tau_{\delta} = \tau$, where $\tau_{\delta}$ is the semi-regularization topology, i.e., the topology on $X$ whose base is the family of all regular open subsets of $(X, \tau)$.

3 Fuzzy upper (lower) δ-precontinuous multifunctions: some characterizations

Definition 3.1
A fuzzy multifunction $F : X \to Y$ is said to be
(a) fuzzy upper δ-precontinuous at a point $x \in X$ if for each fuzzy open set $V$ of $Y$ with $F(x) \subseteq V$, there exists $U \in \delta - PO(X)$ such that $x \in U$, $F(U) \subseteq V$;
(b) fuzzy lower δ-precontinuous at a point $x \in X$ if for each fuzzy open set $V$ of $Y$ with $F(x) \cap V$, there exists $U \in \delta - PO(X)$ such that $x \in U$ and $F(U) \cap V$, for all $u \in U$;
(c) fuzzy upper (lower) δ-precontinuous if $F$ has this property at each point $x \in X$.

Theorem 3.2.
For a fuzzy multifunction $F : X \to Y$, the following statements are equivalent:
(a) $F$ is fuzzy upper δ-precontinuous.
(b) $F^+(V) \in \delta - PO(Y)$ for any fuzzy open set $V$ of $Y$.
(c) $F^-(V)$ is δ-preclosed in $X$ for any fuzzy closed set $V$ of $Y$.
(d) $\delta - pel (F^-(B)) \subseteq F^-(cl B)$ for any fuzzy open set $B$ of $Y$.
(e) $\delta - cl (F^-(B))$ is a δ-preneighbourhood of $x$.
(f) $\delta - cl (F^+(B))$ is a δ-preneighbourhood of $x$.
(g) $F^*(B) \subseteq \delta - pInt (F^+)(B)$ for any fuzzy open set $B$ of $Y$.
(h) $F^*(B) \subseteq \delta - pInt (\delta cl (F^+(B)))$, for any fuzzy open set $B$ of $Y$.
(i) $\delta cl (F^+(V))$ is a nbd of $x$.

Proof. (a) $\Rightarrow$ (b) : Let $V$ be a fuzzy open set of $Y$ and $x \in F^*(V)$. Then $F(x) \subseteq V$. By (a), there exists $U \in \delta - PO(X)$ such that $x \in U$ and $F(U) \subseteq V$. Then $U \subseteq F^+(V)$. Again $U \in \delta - PO(X) \Rightarrow U \subseteq \delta - cl (\delta cl (F^+(V))) \Rightarrow x \in \delta - cl (F^+(V))$. We have $F^+(V) \subseteq \delta - cl (\delta cl (F^+(V)))$ and therefore $F^+(V) \subseteq \delta - PO(X)$.

(b) $\Leftrightarrow$ (c) : Follows from the fact that $F^*(1_Y \setminus V) = X \setminus F^-(V)$, for any fuzzy set $V$ of $Y$.

(c) $\Rightarrow$ (d) : For any fuzzy set $B$ of $Y$, $cl B$ is fuzzy closed in $Y$. By (c), $F^-(cl B)$ is δ-preclosed in $X$. Hence $\delta - pel (F^-(cl B)) \subseteq F^-(cl B)$.

(d) $\Rightarrow$ (e) : Let $V$ be any closed set in $Y$. Then $V = cl V$. So by (d), $\delta - pel (F^-(cl V)) \subseteq F^-(cl V)$ and hence $F^-(V)$ is δ-preclosed in $X$.

(b) $\Rightarrow$ (e) : Let $x \in X$ and $V$ be a fuzzy nbd of $F(x)$. Then there exists a fuzzy open set $G$ in $Y$ such that $F(x) \subseteq G \subseteq V$. Then $x \in F^*(G) \subseteq F^+(V)$ and since $F^+(G) \in \delta - PO(X)$ (by (b)), $F^+(V)$ is a δ-prenebd of $x$. 

Theorem 3.3.

For a fuzzy multifunction \( F : X \to Y \), the following statements are equivalent:

(a) \( F \) is fuzzy lower \( \delta \)-precontinuous.
(b) \( F^+(V) \in \delta - PO(X) \), for any fuzzy open set \( V \) of \( Y \).
(c) \( F^+(V) \) is \( \delta \)-preclosed in \( X \) for any fuzzy closed set \( V \) of \( Y \).
(d) \( \delta - pcl (F^+(B)) \subseteq \delta - pcl (F^+(cl B)) \), for any fuzzy set \( B \) of \( Y \).
(e) For each \( x \in X \) and each fuzzy \( q \)-nbhd \( V \) of \( F(x) \), \( F^{-}(V) \) is a \( \delta \)-prend of \( x \).
(f) For each \( x \in X \) and each fuzzy \( q \)-nbhd \( V \) of \( F(x) \), there exists a \( \delta \)-preend \( U \) of \( x \) such that \( F(U) \subseteq V \).
(g) \( \delta cl (F^{-}(V)) \subseteq \delta cl (F^{-}(cl V)) \), for any fuzzy set \( V \) of \( Y \).
(h) \( \delta cl (F^{-}(B)) \subseteq \delta cl (F^{-}(cl B)) \), for any fuzzy set \( B \) of \( Y \).
(i) For each \( x \in X \) and each fuzzy \( q \)-nbhd \( V \) of \( F(x) \), \( \delta - pcl (F^{-}(V)) \) is a \( \delta \)-nbhd of \( x \).

Proof. (a) \( \Rightarrow \) (b) : Let \( V \) be any fuzzy open set of \( Y \) and \( x \in F^{-}(V) \). Then \( F(x) \) is a \( \delta \)-nbhd of \( x \). Put \( U = F^+(V) \). Then \( F(U) \subseteq V \).

(b) \( \Rightarrow \) (c) : Let \( V \) be any fuzzy open set of \( Y \) and \( x \in F^{-}(V) \). Then by (d), \( F^+(V) \) is a \( \delta \)-preend of \( x \). Put \( U = F^+(V) \). Then \( F(U) \subseteq V \).

(c) \( \Rightarrow \) (d) : For any fuzzy set \( B \) of \( Y \), \( cl B \) is fuzzy closed in \( Y \).

and so by (c), \( F^+(cl B) \) is \( \delta \)-preclosed in \( X \). Therefore, \( \delta - pcl (F^+(B)) \subseteq \delta - pcl (F^+(cl B)) \subseteq F^+(cl B) \).

(d) \( \Rightarrow \) (e) : Let \( V \) be a fuzzy closed set of \( Y \). Then \( \delta - pcl (F^+(V)) \subseteq \delta - pcl (F^+(cl V)) = F^+(cl V) \) which shows that \( F^+(V) \) is \( \delta \)-preclosed in \( X \).

(e) \( \Rightarrow \) (f) : Let \( x \in X \) and \( U \) be a fuzzy \( q \)-nbhd of \( F(x) \). Then by (e), \( F^{-}(V) \) is a \( \delta \)-preend of \( x \). Put \( U = F^{-}(V) \). Then \( F(U) \subseteq V \).

(f) \( \Rightarrow \) (a) : Let \( V \) be any fuzzy open set of \( Y \) such that \( F(x) \subseteq V \). Then \( V \) is a fuzzy \( q \)-nbhd of \( F(x) \). By (f), there exists a \( \delta \)-prend \( U \) of \( x \) such that \( F(U) \subseteq V \). Hence, there exists \( W \in \delta - PO(X) \) such that \( x \in W \subseteq U \) and hence \( F(W) \subseteq F(U) \subseteq V \).

(b) \( \Rightarrow \) (g) : Let \( B \) be any fuzzy set in \( X \). By (b), \( F^+(int B) \in \delta - PO(X) \). Hence \( F^+(int B) \subseteq \delta - pint (F^+(B)) \).

(g) \( \Rightarrow \) (b) : Let \( V \) be any fuzzy open set in \( Y \). Then by (g), \( F^+(V) = F^+(int V) \subseteq \delta - pint (F^+(V)) \) and hence \( F^+(V) \subseteq \delta - PO(X) \).

(b) \( \Leftrightarrow \) (h) : It follows from the definition of \( \delta \)-preopen set in \( X \).

(h) \( \Rightarrow \) (i) : Let \( x \in X \) and \( U \) be a fuzzy \( q \)-nbhd of \( F(x) \). Then there exists a fuzzy open set \( V \) in \( Y \) such that \( F(x) \subseteq V \subseteq U \). Then \( x \in F^{-}(V) \subseteq int (\delta cl (F^{-}(V))) \) and hence \( \delta cl (F^{-}(V)) \) is a \( \delta \)-nbhd of \( x \).

(i) \( \Rightarrow \) (h) : Let \( V \) be any fuzzy open set in \( Y \) and \( x \in F^{-}(V) \). By (i), \( \delta cl (F^{-}(V)) \) is a \( \delta \)-nbhd of \( x \) and thus \( x \in int (\delta cl (F^{-}(V))) \). Hence \( F^{-}(V) \subseteq int (\delta cl (F^{-}(V))) \).

Definition 3.4. [9]

For a fuzzy multifunction \( F : X \to Y \), the fuzzy graph multifunction \( G_F : X \to X \times Y \) of \( F \) is defined as \( G_F(x) = \{ x \times F(x) \} \) of \( F(x) \) of \( X \times Y \), where \( x \times F(x) \) is the fuzzy set in \( X \), whose value is 1 at \( x \in X \) and 0 at other points of \( X \). We shall write \( (x) \times F(x) \) for \( x \times F(x) \).

Lemma 3.5. [2]

The following hold for a fuzzy multifunction \( F : X \to Y \):

(a) \( (G_F)^+(A \times B) = A \cap F^+(B) \)
(b) \( (G_F)^-(A \times B) = A \cap F^-(B) \) for every set \( A \subseteq X \) and fuzzy set \( B \) of \( Y \).

Theorem 3.6.

Let \( X \) be a semi-regular space. Then a fuzzy multifunction
When \( x \in X \) and \( V \) be any fuzzy open set in \( X \times Y \) such that \( F(x) \subseteq V \). Then there exists \( y \in Y \) such that \( [F(x)](y) + V(y) \geq 1 \). Now let \( \gamma \in X \times Y \) such that \( F(x)(\gamma)(y) \in V \). Then \( F(x)(\gamma) \subseteq V \). Since \( F(x) \) is fuzzy lower \( \delta \)-precontinuous, there exists \( U \in \delta - PO(X) \) such that \( F(x)(\gamma) \subseteq U \). By Lemma 2.9, \( U \in \delta - PO(X) \) and \( F(U) \) is fuzzy lower \( \delta \)-precontinuous.

**Theorem 3.9.**

Let \( U_{\alpha} = (\alpha \in \Lambda) \) be a \( \delta \)-open cover of \( X \). A fuzzy multifunction \( F : X \rightarrow Y \) is fuzzy lower \( \delta \)-precontinuous iff the restriction \( F_{\alpha} : X \rightarrow Y \) is fuzzy lower \( \delta \)-precontinuous for each \( \alpha \in \Lambda \).

**Proof.** Let \( \alpha \in \Lambda \) and \( x \in U_{\alpha} \). Let \( V \) be any fuzzy open set in \( Y \) such that \( F(x) \subseteq V \). Since \( \bigcup_{U_{\alpha}} F_{\alpha} : X \rightarrow Y \) is fuzzy upper \( \delta \)-precontinuous, there exists \( U \in \delta - PO(U_{\alpha}) \) with \( x \in U \) such that \( \bigcup_{U_{\alpha}} F_{\alpha} \subseteq U \). Hence \( \bigcup_{U_{\alpha}} F_{\alpha} \) is fuzzy upper \( \delta \)-precontinuous.

**Definition 3.10.**

For a fuzzy multifunction \( F : X \rightarrow Y \), fuzzy multifunction \( \delta - pcl F : X \rightarrow Y \) is given by \( \delta - pcl F(x) = \delta - pcl F(x) \), for each \( x \in X \).

**Theorem 3.11.**

Let \( F : X \rightarrow Y \) be a fuzzy multifunction. Then we have \( (\delta - pcl F)(x) \subseteq \delta - pcl F(x) \) for each \( x \in X \).

**Proof.** Suppose that \( G \) is any fuzzy \( \delta \)-preopen set in \( Y \). Let \( x \in (\delta - pcl F)(x) \). Then \( (\delta - pcl F)(x) q G \Rightarrow F(x) q G \). If not, then \( F(x)(y) + G(y) \geq 1 \). For all \( y \in Y \), \( F(x)(y) \subseteq G \). Therefore, we obtain \( x \in \bigcap_{U_{\alpha}} F_{\alpha} \) and \( (\delta - pcl F)(x) \subseteq \delta - pcl F(x) \). Converse is obvious.

**Theorem 3.12.**

A fuzzy multifunction \( F : X \rightarrow Y \) is fuzzy lower \( \delta \)-precontinuous iff \( \delta - pcl F : X \rightarrow Y \) is fuzzy lower \( \delta \)-precontinuous.

**Proof.** Suppose that \( F \) is fuzzy lower \( \delta \)-precontinuous. Let \( x \in X \) and \( G \) be any fuzzy open set in \( Y \) such that \( (\delta - pcl F)(x) q G \). By Lemma 3.11, \( x \in (\delta - pcl F)(x) \). Therefore, \( (\delta - pcl F)(x) \subseteq \delta - pcl F(x) \). Hence \( F(x) q G \). By fuzzy lower \( \delta \)-precontinuity of \( F \), there exists \( U \in \delta - PO(X) \) with \( x \in U \) such that \( F(x) q G \).
u ∈ U. Since G is fuzzy δ-preopen, by Lemma 3.11, u ∈ F−(G) = (δ − pcl F)(G), for all u ∈ U. Thus (δ − pcl F)(u|G), for all u ∈ U ⇒ δ − pcl F is fuzzy lower δ-precontinuous.

Conversely, suppose that δ − pcl F is fuzzy lower δ-precontinuous. Let x ∈ X and G be any fuzzy open set in Y such that F(x) ∈ G. By Lemma 3.11, we have x ∈ F−(G) = (δ − pcl F)(G) (as fuzzy open sets are fuzzy δ-preopen) and hence (δ − pcl F)(x) ∈ G. Since δ − pcl F is fuzzy lower δ-precontinuous, there exists U ∈ δ − PO(X) with x ∈ U such that (δ − pcl F)(u|G), for all u ∈ U. Since G is fuzzy δ-preopen in Y, by Lemma 3.11, u ∈ (δ − pcl F)(G) = F−(G), for all u ∈ U. Therefore, F(u)|G, for all u ∈ U and hence F is fuzzy lower δ-precontinuous.

Lemma 3.13.
Let F : X → Y be a fuzzy multifunction. Then we have (δ − pcl F)(G) = F+(G), for each G ∈ δ − PO(Y).

Proof. Suppose that G is fuzzy δ-preopen in Y. Let x ∈ (δ − pcl F)(G). Then (δ − pcl F)(x) ≤ G ⇒ F(x) ≤ G ⇒ x ∈ F+(G).
Converse is obvious.

Theorem 3.14.
A fuzzy multifunction F : X → Y is fuzzy upper δ-precontinuous iff δ − pcl F : X → Y is so.

Proof. Suppose that F is fuzzy upper δ-precontinuous. Let x ∈ X and G be any fuzzy open set in Y such that (δ − pcl F)(x) ≤ G. Then F(x) ≤ G. Since F is fuzzy upper δ-precontinuous, there exists U ∈ δ − PO(X) with x ∈ U such that F(U) ≤ G ⇒ U ∈ F+(G) = (δ − pcl F)(G) (as G being fuzzy δ-preopen in ) by Lemma 3.13. Then (δ − pcl F)(U) ≤ G. Hence δ − pcl F is fuzzy upper δ-precontinuous.

Conversely, suppose that δ − pcl F is fuzzy upper δ-precontinuous. Let x ∈ X and G be any fuzzy open set in Y such that F(x) ≤ G ⇒ x ∈ F+(G). As fuzzy open sets are fuzzy δ-preopen, we have by Lemma 3.13, x ∈ (δ − pcl F)(G), i.e., (δ − pcl F)(x) ≤ G. As δ − pcl F is fuzzy upper δ-precontinuous, there exists U ∈ δ − PO(X) with x ∈ U such that (δ − pcl F)(U) ≤ G. Then F(U) ≤ G. Hence F is fuzzy upper δ-precontinuous.

4 CHARACTERIZATIONS OF FUZZY UPPER (LOWER) δ-PRECONTINUITY BY FUZZY UPPER (LOWER)-NBDS

In this section we introduce fuzzy upper and lower nbds of a fuzzy set and characterize fuzzy upper and lower δ-precontinuity via these new concepts.

Definition 4.1.
A fuzzy set A in an fts Y is said to be a fuzzy lower (upper) nbd of a fuzzy set B in Y if there exists a fuzzy open set V in Y such that B ∈ V (B ≤ V) and V ∊ (1 \ A).

Theorem 4.2.
A fuzzy multifunction F : X → Y is fuzzy lower δ-precontinuous iff for each x ∈ X and each fuzzy lower nbd M of F(x), F−(M) is a δ-prenbd of x.

Proof. Let F be fuzzy lower δ-precontinuous. Let x ∈ X and M be a fuzzy lower nbd of F(x). Then there exists a fuzzy open set V in Y such that x ∈ F−(V) and V ∊ (1 \ M) ⇒ V ≤ M. As F is fuzzy lower δ-precontinuous, there exists U ∈ δ − PO(X) with x ∈ U and U ⊆ F−(V). Then x ∈ U ⊆ F−(V) ⊆ F−(M) and so F−(M) is a δ-prenbd of x.

Conversely, let x ∈ X and V be any fuzzy open set in Y with F(x)q V. Now V ∊ (1 \ V). Then V is a fuzzy lower nbd of F(x).

By the given condition, F−(V) is a δ-prenbd of x. Hence there exists U ∈ δ − PO(X) containing x such that x ∈ U ⊆ F−(V), i.e., F(x)q V, for all x ∈ U. Hence F is fuzzy lower δ-precontinuous.

Theorem 4.3.
A fuzzy multifunction F : X → Y is fuzzy upper δ-precontinuous iff for each x0 ∈ X and each fuzzy upper nbd M of F(x0), F+(M) is a δ-prenbd of x0.

Proof. Let F be fuzzy upper δ-precontinuous. Then for any x0 ∈ X and for any fuzzy upper nbd M of F(x0), there exists a fuzzy open set V in Y such that F(x0) ≤ V and V ∊ (1 \ M), i.e., x0 ∈ F+(V) and V ≤ M. Since F is fuzzy upper δ-precontinuous, there exists U ∈ δ − PO(X) with x0 ∈ U such that F(U) ≤ V ≤ M. Then x0 ∈ U ⊆ F+(V) ⊆ F+(M), showing that F+(M) is a δ-prenbd of x0.

Conversely, for any x0 ∈ X and any fuzzy open set V in Y with x0 ∈ F+(V), we have F(x0) ≤ V and V ∊ (1 \ V). So V is a fuzzy upper nbd of F(x0). Then by the given condition, F+(V) is a δ-prenbd of x0. Then there exists U ∈ δ − PO(X) with x0 ∈ U such that F(U) ≤ V, and hence F is fuzzy upper δ-precontinuous.

5 APPLICATIONS OF FUZZY UPPER (LOWER) δ-PRECONTINUITY

Definition 5.1. [12]
A topological space (X, τ) is said to be δ-preregular if for any x ∈ X and any δ-preopen set V of X containing x, there exists an open set U in X such that x ∈ U ⊆ δ − pcl U ⊆ V.

Theorem 5.2.
Let F : X → Y be a surjective fuzzy multifunction and F(x) be a fuzzy compact set in Y for each x ∈ X. If F is fuzzy upper δ-precontinuous and X is δ-preregular, compact, then Y is fuzzy compact.

Proof. Let (Ax : a ∈ A) be a fuzzy cover of Y by fuzzy open sets in Y. Now for each x ∈ X, F(x) is fuzzy compact in Y, and so there is a finite subset Ax of A such that F(x) ⊆ VAx :
\[ \alpha \in \Lambda x \}. \text{ Let } A_x = \vee \{ A_\alpha : \alpha \in \Lambda x \}. \text{ Then } F(x) \leq A_x \text{ and } A_x \text{ is fuzzy open in } Y. \text{ Since } F \text{ is fuzzy upper } \delta \text{-precontinuous, there exists } U_x \in \delta \text{-PO}(X) \text{ with } x \in U_x \text{ such that } U_x \subseteq F^+(A_x). \text{ As } X \text{ is } \delta \text{-preregular, there exists an open set } V_x \text{ in } X \text{ such that } x \in V_x \subseteq \delta - \text{pcl } V_x \subseteq U_x. \text{ Then the family } \{ V_x : x \in X \} \text{ is a fuzzy open cover of } X. \text{ Then there exist finitely many points } x_1, x_2, ..., x_n \text{ in } X \text{ such that } X = \bigcup_{i=1}^{n} V_{x_i}. \text{ As } F \text{ is surjective, we have } 1_Y = F(X) = F(\bigcup_{i=1}^{n} V_{x_i}) = \bigvee_{i=1}^{n} F(V_{x_i}) \leq \bigvee_{i=1}^{n} A_{x_i} \leq \bigvee_{\alpha \in \Lambda x_i} A_\alpha. \text{ Hence } Y \text{ is fuzzy compact.}

**Definition 5.3.** [8]

An fts \( (Y, \tau_Y) \) is said to be FNC-space if every regular open cover of \( Y \) has a finite subcover.

**Remark 5.4.**

As every fuzzy regular open set is fuzzy open, we get the following theorem.

**Theorem 5.5.**

Let \( F : X \to Y \) be a surjective fuzzy multifunction and \( F(x) \) be a fuzzy compact set in \( Y \) for each \( x \in X \). If \( F \) is fuzzy upper \( \delta \)-precontinuous and \( X \) is \( \delta \)-preregular, compact, then \( Y \) is FNC-space.

**REFERENCES**


