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On New Separation Axioms Via γ-Open Sets*

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ABSTRACT

In this paper, we introduce two new classes of topological spaces called γ-R_0 and γ-R_1 spaces in terms of the concept of γ-open sets and investigate some of their fundamental properties.

Keywords : γ-open, γ-closure, γ-R_0 spaces and γ-R_1 spaces.

1 INTRODUCTION

THE notion of R_0 topological spaces is introduced by Shanin [4] in 1943. Later, Davis [2] rediscovered it and studied some properties of this weak separation axiom. In the same paper, Davis also introduced the notion of R_1 topological space which are independent of both T_0 and T_1 but strictly weaker than T_2. The notion of γ-open sets was introduced by Ogata [3]. In this paper, we continue the study of the above mentioned classes of topological spaces satisfying these axioms by introducing two more notions in terms of γ-open sets called γ-R_0 and γ-R_1.

2 Preliminaries

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y) denotes a topological spaces on which no separation axioms is assumed unless explicitly stated. Let A be a subset of a topological space X. The closure of A is denoted by Cl(A).

Definition 2.1. [3] Let (X, τ) be a topological space. An operation γ on the topology τ is a mapping from τ in to power set P(X) of X such that $V \subseteq \gamma(V)$ for each $V \in \tau$, where $\gamma(V)$ denotes the value of γ at V.

Definition 2.2. [3] A subset A of a topological spac (X, τ) is called γ-open set if for each $x \in A$ there exists an open set U such that $x \in U$ and $\gamma(U) \subseteq A$. Complements of γ-open sets are called γ-closed.

$\gamma O(X)$ denotes the collection of all γ-open sets of (X, τ). Moreover, $\gamma C(X)$ denotes the collection of all γ-closed sets of (X, τ).

Definition 2.3. [1] A γ-nbd of $x \in X$ is a set U of X which contains a γ-open set V containing x.

Definition 2.4. [3] The intersection of all γ-closed sets containing A is called the γ-closure of A and is denoted by τ_γ-Cl(A).

3 γ-R_0 and γ-R_1 spaces

We introduce the following definitions.

Definition 3.1. Let A be a subset of a topological space (X, τ) and γ be an operation on τ. The γ-kernel of A, denoted by $\gamma\ker(A)$ is defined to be the set

$$\gamma\ker(A) = \cap \{U \in \gamma O(X): A \subseteq U\}.$$

Lemma 3.2. Let (X, τ) be a topological space with an operation γ on τ and $x \in X$. Then $y \in \gamma\ker(\{x\})$ if and only if $x \in \tau_\gamma$-Cl($\{y\}$).

Proof. Suppose that $y \notin \gamma\ker(\{x\})$. Then there exists a γ-open set V containing x such that $y \notin V$. Therefore, we have $x \notin \tau_\gamma$-Cl($\{y\}$). The proof of the converse case can be done similarly.

Theorem 3.3. Let (X, τ) be a topological space with an operation γ on τ and A be a subset of X. Then, $\gamma\ker(A) = \{x \in X: \tau_\gamma\text{-Cl}(\{x\}) \cap A \neq \phi\}$.

Proof. Let $x \in \gamma\ker(A)$ and suppose τ_γ-Cl($\{x\}$) $\cap A = \phi$. Hence $x \notin X \setminus \tau_\gamma$-Cl($\{x\}$) which is a γ-open set containing A. This is impossible, since $x \in \gamma\ker(A)$. Consequently, τ_γ-Cl($\{x\}$) $\cap A \neq \phi$. Next, let $x \in X$ such that τ_γ-Cl($\{x\}$) $\cap A \neq \phi$ and suppose that $x \notin \gamma\ker(A)$. Then, there exists a γ-open set V containing A and $x \notin V$. Let $y \in \tau_\gamma$-Cl($\{x\}$) $\cap$ A. Hence, V is a γ-nbd of y which does not contain x. By this contradiction $x \in \gamma\ker(A)$ and the claim.

Theorem 3.4. The following properties hold for the subsets A, B of a topological space (X, τ) with an operation γ on τ:

1. $A \subseteq \gamma\ker(A)$.
2. $A \subseteq B$ implies that $\gamma\ker(A) \subseteq \gamma\ker(B)$.
3. If A is γ-open in (X, τ), then $A = \gamma\ker(A)$.
4. $\gamma\ker(\gamma\ker(A)) = \gamma\ker(A)$.

Proof. (1), (2) and (3) are immediate consequences of Definition 3.1. To prove (4), first observe that by (1) and (2), we have $\gamma\ker(A) \subseteq \gamma\ker(\gamma\ker(A))$. If $x \notin \gamma\ker(A)$, then there exists $U \in \gamma O(X)$ such that $A \subseteq U$ and $x \notin U$. Hence $\gamma\ker(A) \subseteq U$, and so we have $x \notin \gamma\ker(\gamma\ker(A))$. Thus $\gamma\ker(\gamma\ker(A)) = \gamma\ker(A)$.

Definition 3.5. A topological space (X, τ) with an operation operation γ on τ, is said to be γ-R_0 if U is a γ-open set and $x \in U$ then τ_γ-Cl($\{x\}$) $\subseteq$ U.

Theorem 3.6. For a topological space (X, τ) with an operation γ operation γ on τ, the following properties are equivalent:

1. (X, τ) is γ-R_0.
2. For any $F \in \gamma C(X)$, $x \notin F$ implies $F \subseteq U$ and $x \notin U$ for

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some $U \in \gamma O(X)$.

3. For any $F \in \gamma C(X)$, $x \notin F$ implies $F \cap \tau_\gamma\text{-Cl}(\{x\}) = \phi$.
4. For any distinct points x and y of X, either $\tau_\gamma\text{-Cl}(\{x\}) = \tau_\gamma\text{-Cl}(\{y\})$ or $\tau_\gamma\text{-Cl}(\{x\}) \cap \tau_\gamma\text{-Cl}(\{y\}) = \phi$.

Proof. (1) ⇒ (2). Let $F \in \gamma C(X)$ and $x \notin F$. Then by (1), $\tau_\gamma\text{-Cl}(\{x\}) \subseteq X \setminus F$. Set $U = X \setminus \tau_\gamma\text{-Cl}(\{x\})$, then U is a γ-open set such that $F \subseteq U$ and $x \notin U$.
(2) ⇒ (3). Let $F \in \gamma C(X)$ and $x \notin F$. There exists $U \in \gamma O(X)$ such that $F \subseteq U$ and $x \notin U$. Since $U \in \gamma O(X)$, $U \cap \tau_\gamma\text{-Cl}(\{x\}) = \phi$ and $F \cap \tau_\gamma\text{-Cl}(\{x\}) = \phi$.
(3) ⇒ (4). Suppose that $\tau_\gamma\text{-Cl}(\{x\}) \neq \tau_\gamma\text{-Cl}(\{y\})$ for distinct points $x, y \in X$. There exists $z \in \tau_\gamma\text{-Cl}(\{x\})$ such that $z \notin \tau_\gamma\text{-Cl}(\{y\})$ (or $z \in \tau_\gamma\text{-Cl}(\{y\})$ such that $z \notin \tau_\gamma\text{-Cl}(\{x\})$). There exists $V \in \gamma O(X)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin \tau_\gamma\text{-Cl}(\{y\})$. By (3), we obtain $\tau_\gamma\text{-Cl}(\{x\}) \cap \tau_\gamma\text{-Cl}(\{y\}) = \phi$.
(4) ⇒ (1). let $V \in \gamma O(X)$ and $x \in V$. For each $y \notin V$, $x \neq y$and $x \notin \tau_\gamma\text{-Cl}(\{y\})$. This shows that $\tau_\gamma\text{-Cl}(\{x\}) \neq \tau_\gamma\text{-Cl}(\{y\})$. By (4), $\tau_\gamma\text{-Cl}(\{y\}) = \phi$ for each $y \in X\setminus V$ and hence $\tau_\gamma\text{-Cl}(\{x\}) \cap (\cup_{y\in X\setminus V} \tau_\gamma\text{-Cl}(\{y\})) = \phi$. On other hand, since $V \in \gamma O(X)$ and $y \in X\setminus V$, we have $\tau_\gamma\text{-Cl}(\{y\}) \subseteq X \setminus V$ and hence $X \setminus V = \cup_{y\in X\setminus V} \tau_\gamma\text{-Cl}(\{y\})$. Therefore, we obtain $(X \setminus V) \cap \tau_\gamma\text{-Cl}(\{x\}) = \phi$ and $\tau_\gamma\text{-Cl}(\{x\}) \subseteq V$. This shows that (X, τ) is a γ-R_0 space.

Theorem 3.7. For a topological space (X, τ) with an operation γ on τ, the following properties are equivalent:

1. (X, τ) is γ-R_0.
2. $x \in \tau_\gamma\text{-Cl}(\{y\})$ if and only if $y \in \tau_\gamma\text{-Cl}(\{x\})$, for any points x and y in X.

Proof. (1) ⇒ (2). Assume that X is γ-R_0. Let $x \in \tau_\gamma\text{-Cl}(\{y\})$ and V be any γ-open set such that $y \in V$. Now by hypothesis, $x \in V$. Therefore, every γ-open set which contain y contains x. Hence $y \in \tau_\gamma\text{-Cl}(\{x\})$.
(2) ⇒ (1). Let U be a γ-open set and $x \in U$. If $y \notin U$, then $x \notin \tau_\gamma\text{-Cl}(\{y\})$ and hence $y \notin \tau_\gamma\text{-Cl}(\{x\})$. This implies that $\tau_\gamma\text{-Cl}(\{x\}) \subseteq U$. Hence (X, τ) is γ-R_0.

Theorem 3.8. The following statements are equivalent for any points x and y in a topological space (X, τ) with an operation γ on τ:

1. $\gamma\ker(\{x\}) \neq \gamma\ker(\{y\})$.
2. $\tau_\gamma\text{-Cl}(\{x\}) \neq \tau_\gamma\text{-Cl}(\{y\})$.

Proof. (1) ⇒ (2). Suppose that $\gamma\ker(\{x\}) \neq \gamma\ker(\{y\})$, then there exists a point z in X such that $z \in \gamma\ker(\{x\})$ and $z \notin \gamma\ker(\{y\})$. From $z \in \gamma\ker(\{x\})$ it follows that $\{x\} \cap \tau_\gamma\text{-Cl}(\{z\}) \neq \phi$ which implies $x \in \tau_\gamma\text{-Cl}(\{z\})$. By $z \notin \gamma\ker(\{y\})$, we have $\{y\} \cap \tau_\gamma\text{-Cl}(\{z\}) = \phi$. Since $x \in \tau_\gamma\text{-Cl}(\{z\})$, $\tau_\gamma\text{-Cl}(\{x\}) \subseteq \tau_\gamma\text{-Cl}(\{z\})$ and $\{y\} \cap \tau_\gamma\text{-Cl}(\{x\}) = \phi$. Therefore, it follows that $\tau_\gamma\text{-Cl}(\{x\}) \neq \tau_\gamma\text{-Cl}(\{y\})$. Now $\gamma\ker(\{x\}) \neq \gamma\ker(\{y\})$ implies that $\tau_\gamma\text{-Cl}(\{x\}) \neq \tau_\gamma\text{-Cl}(\{y\})$.
(2) ⇒ (1). Suppose that $\tau_\gamma\text{-Cl}(\{x\}) \neq \tau_\gamma\text{-Cl}(\{y\})$. Then there exists a point z in X such that $z \in \tau_\gamma\text{-Cl}(\{x\})$ and $z \notin \tau_\gamma\text{-Cl}(\{y\})$. Then, there exists a γ-open set containing z and therefore x but not y, namely, $y \notin \gamma\ker(\{x\})$ and thus $\gamma\ker(\{x\}) \neq \gamma\ker(\{y\})$.

Theorem 3.9. Let(X, τ) be a topological space and γ be an operation on τ. Then $\cap \{\tau_\gamma\text{-Cl}(\{x\}) : x \in X\} = \phi$ if and only if $\gamma\ker(\{x\}) \neq X$ for every $x \in X$.

Proof. Necessity. Suppose that $\cap\{\tau_\gamma\text{-Cl}(\{x\}) : x \in X\} = \phi$. Assume that there is a point y in X such that $\gamma\ker(\{y\}) = X$. Let x be any point of X. Then $x \in V$ for every γ-open set V containing y and hence $y \in \tau_\gamma\text{-Cl}(\{x\})$ for any $x \in X$. This implies that $y \in \cap \{\tau_\gamma\text{-Cl}(\{x\}) : x \in X\}$. But this is a contradiction.
Sufficiency. Assume that $\gamma\ker(\{x\}) \neq X$ for every $x \in X$. If there exists a point y in X such that $y \in \cap \{\tau_\gamma\text{-Cl}(\{x\}) : x \in X\}$, then every γ-open set containing y must contain every point of X. This implies that the space X is the unique γ-open set containing y. Hence $\gamma\ker(\{y\}) = X$ which is a contradiction. Therefore, $\cap \{\tau_\gamma\text{-Cl}(\{x\}) : x \in X\} = \phi$.

Theorem 3.10. A topological space (X, τ) with an operation γ on τ is γ-R_0 if and only if for every x and y in X,
$\tau_\gamma\text{-Cl}(\{x\}) \neq \tau_\gamma\text{-Cl}(\{y\})$ implies $\tau_\gamma\text{-Cl}(\{x\}) \cap \tau_\gamma\text{-Cl}(\{y\}) = \phi$.

Proof. Necessity. Suppose that (X, τ) is γ-R_0 and $\tau_\gamma\text{-Cl}(\{x\}) \neq \tau_\gamma\text{-Cl}(\{y\})$. Then, there exists $z \in \tau_\gamma\text{-Cl}(\{x\})$ such that $z \notin \tau_\gamma\text{-Cl}(\{y\})$ (or $z \in \tau_\gamma\text{-Cl}(\{y\})$ such that $z \notin \tau_\gamma\text{-Cl}(\{x\})$). There exists $V \in \gamma O(X)$ such that $y \notin V$ and $z \in V$, hence $x \in V$. Therefore, we have $x \notin \tau_\gamma\text{-Cl}(\{y\})$. Thus $x \in [X \setminus \tau_\gamma\text{-Cl}(\{y\})] \in \gamma O(X)$, which implies $\tau_\gamma\text{-Cl}(\{x\}) \subseteq [X \setminus \tau_\gamma\text{-Cl}(\{y\})]$ and $\tau_\gamma\text{-Cl}(\{x\}) \cap \tau_\gamma\text{-Cl}(\{y\}) = \phi$.
Sufficiency. Let $V \in \gamma O(X)$ and let $x \in V$. We still show that $\tau_\gamma\text{-Cl}(\{x\}) \subseteq V$. Let $y \notin V$, that is $y \in X\setminus V$. Then $x \neq y$ and $x \notin \tau_\gamma\text{-Cl}(\{y\})$. This shows that $\tau_\gamma\text{-Cl}(\{x\}) \neq \tau_\gamma\text{-Cl}(\{y\})$. By assumption, $\tau_\gamma\text{-Cl}(\{x\}) \cap \tau_\gamma\text{-Cl}(\{y\}) = \phi$. Hence $y \notin \tau_\gamma\text{-Cl}(\{x\})$ and therefore $\tau_\gamma\text{-Cl}(\{x\}) \subseteq V$.

Theorem 3.11. A topological space (X, τ) with an operation γ on τ is γ -R_0 if and only if for any points x and y in X, $\gamma\ker(\{x\}) \neq \gamma\ker(\{y\})$ implies $\gamma\ker(\{x\}) \cap \gamma\ker(\{y\}) = \phi$.

Proof. Suppose that (X, τ) is a γ-R_0 space. Thus by Theorem 3.8, for any points x and y in X if $\gamma\ker(\{x\}) \neq \gamma\ker(\{y\})$ then $\tau_\gamma\text{-Cl}(\{x\}) \neq \tau_\gamma\text{-Cl}(\{y\})$. Now we prove that $\gamma\ker(\{x\}) \cap \gamma\ker(\{y\}) = \phi$. Assume that $z \in \gamma\ker(\{x\}) \cap \gamma\ker(\{y\})$. By $z \in \gamma\ker(\{x\})$ and Lemma 3.2, it follows that $x \in \tau_\gamma\text{-Cl}(\{z\})$. Since $x \in \tau_\gamma\text{-Cl}(\{x\})$, by Theorem 3.6, $\tau_\gamma\text{-Cl}(\{x\}) = \tau\gamma\text{-Cl}(\{z\})$. Similarly, we have $\tau_\gamma\text{-Cl}(\{y\}) = \tau_\gamma\text{-Cl}(\{z\}) = \tau_\gamma\text{-Cl}(\{x\})$. This is a contradiction. Therefore, we have $\gamma\ker(\{x\}) \cap \gamma\ker(\{y\}) = \phi$.
Conversely, let (X, τ) be a topological space such that for any points x and y in X, $\gamma\ker(\{x\}) \neq \gamma\ker(\{y\})$ implies $\gamma\ker(\{x\}) \cap \gamma\ker(\{y\}) = \phi$. If $\tau_\gamma\text{-Cl}(\{x\}) \neq \tau_\gamma\text{-Cl}(\{y\})$, then by Theorem 3.8, $\gamma\ker(\{x\}) \neq \gamma\ker(\{y\})$. Hence, $\gamma\ker(\{x\}) \cap \gamma\ker(\{y\}) = \phi$ which implies $\tau_\gamma\text{-Cl}(\{x\}) \cap \tau_\gamma\text{-Cl}(\{y\}) = \phi$. Because $z \in \tau_\gamma\text{-Cl}(\{x\})$ implies that $x \in \gamma\ker(\{z\})$ and therefore $\gamma\ker(\{x\}) \cap \gamma\ker(\{z\}) \neq \phi$. By hypothesis, we have $\gamma\ker(\{x\}) = \gamma\ker(\{z\})$. Then $z \in \tau_\gamma\text{-Cl}(\{x\}) \cap \tau_\gamma\text{-Cl}(\{y\})$ implies that $\gamma\ker(\{x\}) = \gamma\ker(\{z\}) = \gamma\ker(\{y\})$. This is a contradiction. Therefore, $\tau_\gamma\text{-Cl}(\{x\}) \cap \tau_\gamma\text{-Cl}(\{y\}) = \phi$ and by Theorem 3.6, (X, τ) is a γ-R_0 space.

Theorem 3.12. For a topological space (X, τ) with an operation operation γ on τ, the following properties are equivalent:

1. (X, τ) is a γ-R_0 space.
2. For any non-empty set there exists $F \in \gamma C(X)$ such that $A \cap F \neq \phi$ and $F \subseteq G$.

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3. For any $G \in \gamma O(X)$, we have $G = \cup \{F \in \gamma C(X): \subseteq G\}$.
4. For any $F \in \gamma C(X)$, we have $F = \cap \{G \in \gamma O(X): F \subseteq G\}$.
5. For every $x \in X$, $\tau_\gamma\text{-}Cl(\{x\}) \subseteq \gamma ker(\{x\})$.

Proof. (1) ⇒ (2). Let A be a non-empty subset of X and $G \in \gamma O(X)$ such that $A \cap G \neq \phi$. There exists $x \in A \cap G$. Since $x \in G \in \gamma O(X)$, $\tau_\gamma\text{-}Cl(\{x\}) \subseteq G$. Set $F = \tau_\gamma\text{-}Cl(\{x\})$, then $F \in \gamma C(X)$, $F \subseteq G$ and $A \cap F \neq \phi$.

(2) ⇒ (3). Let $G \in \gamma O(X)$, then $G \supseteq \cup \{F \in \gamma C(X): F \subseteq G\}$. Let x be any point of G. There exists $F \in \gamma C(X)$ such that $x \in F$ and $F \subseteq G$. Therefore, we have $x \in F \subseteq \cup \{F \in \gamma C(X): F \subseteq G\}$ and hence $G = \cup \{F \in \gamma C(X): F \subseteq G\}$.

(3) ⇒ (4). Obvious.

(4) ⇒ (5). Let x be any point of X and $y \notin \gamma ker(\{x\})$. There exists $V \in \gamma O(X)$ such that $x \in V$ and $y \notin V$, hence $\tau_\gamma\text{-}Cl(\{y\}) \cap V = \phi$. By (4), $(\cap \{G \in \gamma O(X): \tau_\gamma\text{-}Cl(\{y\}) \subseteq G\}) \cap V = \phi$ and there exists $G \in \gamma O(X)$ such that $x \notin G$ and $\tau_\gamma\text{-}Cl(\{y\}) \subseteq G$. Therefore $\tau_\gamma\text{-}Cl(\{x\}) \cap G = \phi$ and $y \notin \tau_\gamma\text{-}Cl(\{x\})$. Consequently, we obtain $\tau_\gamma\text{-}Cl(\{x\}) \subseteq \gamma ker(\{x\})$.

(5) ⇒ (1). Let $G \in \gamma O(X)$ and $x \in G$. Let $y \in \gamma ker(\{x\})$, then $x \in \tau_\gamma\text{-}Cl(\{y\})$ and $y \in G$. This implies that $\gamma ker(\{x\}) \subseteq G$. Therefore, we obtain $x \in \tau_\gamma\text{-}Cl(\{x\}) \subseteq \gamma ker(\{x\}) \subseteq G$. This shows that (X, τ) is a γ-R_0 space.

Corollary 3.13. For a topological space (X, τ) with an operation γ on τ, the following properties are equivalent:

1. (X, τ) is a γ-R_0 space.
2. $\tau_\gamma\text{-}Cl(\{x\}) = \gamma ker(\{x\})$ for all $x \in X$.

Proof. (1) ⇒ (2). Suppose that (X, τ) is a γ-R_0 space. By Theorem 3.12, $\tau_\gamma\text{-}Cl(\{x\}) \subseteq \gamma ker(\{x\})$ for each $x \in X$. Let $y \in \gamma ker(\{x\})$, then $x \in \tau_\gamma\text{-}Cl(\{y\})$ and by Theorem 3.6, $\tau_\gamma\text{-}Cl(\{x\}) = \tau_\gamma\text{-}Cl(\{y\})$. Therefore, $y \in \tau_\gamma\text{-}Cl(\{x\})$ and hence $\gamma ker(\{x\}) \subseteq \tau_\gamma\text{-}Cl(\{x\})$. This shows that $\tau_\gamma\text{-}Cl(\{x\}) = \gamma ker(\{x\})$.

(2) ⇒ (1). Follows from Theorem 3.12.

Theorem 3.14. For a topological space (X, τ) with an operation γ on τ, the following properties are equivalent:

1. (X, τ) is a γ-R_0 space.
2. If F is γ-closed, then $F = \gamma ker(F)$.
3. If F is γ-closed and $x \in F$, then $\gamma ker(\{x\}) \subseteq F$.
4. If $x \in X$, then $\gamma ker(\{x\}) \subseteq \tau_\gamma\text{-}Cl(\{x\})$.

Proof. (1) ⇒ (2). Let F be a γ-closed and $x \notin F$. Thus $(X\backslash F)$ is a γ-open set containing x. Since (X, τ) is γ-R_0, $\tau_\gamma\text{-}Cl(\{x\}) \subseteq (X\backslash F)$. Thus $\tau_\gamma\text{-}Cl(\{x\}) \cap F = \phi$ and by Theorem 3.3, $x \notin \gamma ker(F)$. Therefore $\gamma ker(F) = F$.

(2) ⇒ (3). In general, $A \subseteq B$ implies $\gamma ker(A) \subseteq \gamma ker(B)$. Therefore, it follows from (2), that $\gamma ker(\{x\}) \subseteq \gamma ker(F) = F$.

(3) ⇒ (4). Since $x \in \tau_\gamma\text{-}Cl(\{x\})$ and $\tau_\gamma\text{-}Cl(\{x\})$ is γ-closed, by (3), $\gamma ker(\{x\}) \subseteq \tau_\gamma\text{-}Cl(\{x\})$.

(4) ⇒ (1). We show the implication by using Theorem 3.7. Let $x \in \tau_\gamma\text{-}Cl(\{y\})$. Then by Lemma 3.2, $y \in \gamma ker(\{x\})$. Since $x \in \tau_\gamma\text{-}Cl(\{x\})$ and $\tau_\gamma\text{-}Cl(\{x\})$ is γ-closed, by (4), we obtain $y \in \gamma ker(\{x\}) \subseteq \tau_\gamma\text{-}Cl(\{x\})$. Therefore $x \in \tau_\gamma\text{-}Cl(\{y\})$ implies $y \in \tau_\gamma\text{-}Cl(\{x\})$. The converse is obvious and (X, τ) is γ-R_0.

Definition 3.15. A topological space (X, τ) with an operation γ on τ, is said to be γ-R_1 if for x, y in X with $\tau_\gamma\text{-}Cl(\{x\}) \neq \tau_\gamma\text{-}Cl(\{y\})$, there exist disjoint γ-open sets U and V such that $\tau_\gamma\text{-}Cl(\{x\}) \subseteq U$ and $\tau_\gamma\text{-}Cl(\{y\}) \subseteq V$.

Theorem 3.16. For a topological space (X, τ) with an operation γ on τ, the following statements are equivalent:

1. (X, τ) is γ-R_1.
2. If $x, y \in X$ such that $\tau_\gamma\text{-}Cl(\{x\}) \neq \tau_\gamma\text{-}Cl(\{y\})$, then there exist γ-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.

Proof. Obvious.

Theorem 3.17. If (X, τ) is γ-R_1, then (X, τ) is γ-R_0.

Proof. Let U be γ-open such that $x \in U$. If $y \notin U$, since $x \notin \tau_\gamma\text{-}Cl(\{y\})$, we have $\tau_\gamma\text{-}Cl(\{x\}) \neq \tau_\gamma\text{-}Cl(\{y\})$. So, there exists a γ-open set V such that $\tau_\gamma\text{-}Cl(\{y\}) \subseteq V$ and $x \notin V$, which implies $y \notin \tau_\gamma\text{-}Cl(\{x\})$. Hence $\tau_\gamma\text{-}Cl(\{x\}) \subseteq U$. Therefore, (X, τ) is γ-R_0.

The converse of the above Theorem need not be ture in general as shown in the following example.

Example 3.18. Consider X = {a, b, c} with the discrete topology on X. Define an operation γ on τ by $\gamma(A) = A$ if A = {a, b} or {a, c} or {b, c} and $\gamma(A) = X$ otherwise. Then X is a γ-R_0 space but not a γ-R_1 space.

Corollary 3.19. A topological space (X, τ) with an operation γ on τ is γ-R_1 if and only if for $x, y \in X$, $\gamma ker(\{x\}) \neq \gamma ker(\{y\})$, there exist disjoint γ-open sets U and V such that $\tau_\gamma\text{-}Cl(\{x\}) \subseteq U$ and $\tau_\gamma\text{-}Cl(\{y\}) \subseteq V$.

Proof. Follows from Theorem 3.8.

Theorem 3.20. A topological space (X, τ) is γ-R_1 if and only if $x \in X\backslash\tau_\gamma\text{-}Cl(\{y\})$ implies that x and y have disjoint γ-nbds.

Proof. Necessity. Let $x \in X\backslash\tau_\gamma\text{-}Cl(\{y\})$. Then $\tau_\gamma\text{-}Cl(\{x\}) \neq \tau_\gamma\text{-}Cl(\{y\})$, so, x and y have disjoint γ-nbds.

Sufficiency. First, we show that (X, τ) is γ-R_0. Let U be a γ-open set and $x \in U$. Suppose that $y \notin U$. Then, $\tau_\gamma\text{-}Cl(\{y\}) \cap U = \phi$ and $x \notin \tau_\gamma\text{-}Cl(\{y\})$. There exist γ-open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \phi$. Hence, $\tau_\gamma\text{-}Cl(\{x\}) \subseteq \tau_\gamma\text{-}Cl(U_x)$ and $\tau_\gamma\text{-}Cl(\{x\}) \cap U_y \subseteq \tau_\gamma\text{-}Cl(U_x) \cap U_y = \phi$. Therefore, $y \notin \tau_\gamma\text{-}Cl(\{x\})$. Consequently, $\tau_\gamma\text{-}Cl(\{x\}) \subseteq U$ and (X, τ) is γ-R_0. Next, we show that (X, τ) is γ-R_1. Suppose that $\tau_\gamma\text{-}Cl(\{x\}) \neq \tau_\gamma\text{-}Cl(\{y\})$. Then, we can assume that there exists $z \in \tau_\gamma\text{-}Cl(\{x\})$ such that $z \notin \tau_\gamma\text{-}Cl(\{y\})$. There exist γ-open sets V_z and V_y such that $z \in V_z$, $y \in V_y$

And $V_z \cap V_y = \phi$. Since $z \in \tau_\gamma\text{-}Cl(\{x\})$, $x \in V_z$. Since (X, τ) is γ-R_0, we obtain $\tau_\gamma\text{-}Cl(\{x\}) \subseteq V_z$, $\tau_\gamma\text{-}Cl(\{y\}) \subseteq V_y$ and $V_z \cap V_y = \phi$. This shows that (X, τ) is γ-R_1.

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