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On Double Integrals

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ABSTRACT

The present paper is related with the study of the integration of certain products of $\overline{H}$-function [7], M-series [9] and a general class of polynomials [11]. During the course of finding, we establish certain new double integration involving to a product of a general class of polynomial, M- series and $\overline{H}$-functions. The results of this paper are believed to be new and basic in nature and are likely to find useful application in various fields notably electrical networks, probability theory and statistical mechanics etc.

Key Words: Feynman integrals, $\overline{H}$-function, M-series, A general class of polynomials.

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1. INTRODUCTION

The $\overline{H}$-function [7] is ageneratization of the well known Fox's H-function [4]. The $\overline{H}$-function pertains to exact partition function of the Gaussian model in statistical mechanics, functions useful in testing hypothesis and several other as its special cases.

The $\overline{H}$-function will be defined as [1]:

$$\overline{H}_{p,q}^{m,n}[z] = \overline{H}_{p,q}^{m,n}\left[z \left| \begin{matrix} (e_j,\gamma_j;E_j)_{1,n},(e_j,\gamma_j)_{n+1,p} \\ (f_j,\delta_j)_{1,m},(f_j,\delta_j;F_j)_{m+1,q} \end{matrix} \right. \right]$$

$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\bar{\phi}(\xi)z^{\xi}\,d\xi \qquad \ldots(1.1)$$

where

$$\bar{\phi}(\xi)=\frac{\prod_{j=1}^{m}\Gamma(f_j-\delta_j\xi)\prod_{j=1}^{n}\{\Gamma(1-e_j+\gamma_j\xi)\}^{E_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-f_j+\delta_j\xi)\}^{F_j}\prod_{j=n+1}^{p}\Gamma(e_j-\gamma_j\xi)} \qquad \ldots(1.2)$$

which contains fractional powers of some of the gamma functions. Here and throughout the paper e_j (j = 1,…,p) and f_j (j = 1,…,q) are complex parameters, $\gamma_j \geq 0\,(j=1,...,q), \delta_j\,(j=1,...,q)$ (not all zero simultaneously) and the exponents E_j (j = 1,…,n) and F_j (j=m+1,…,q) can take non-integer values.

The $\bar{H}$-function in (1) converges absolutely if

$$|\arg(z)|<\frac{1}{2}\Omega\,\pi \qquad \ldots(1.3)$$

where

$$\Omega=\sum_{i=1}^{m}\delta_j+\sum_{j=1}^{n}E_j\gamma_j-\sum_{j=m+1}^{q}F_j\delta_j-\sum_{j=n+1}^{p}\gamma_j>0 \qquad \ldots(1.4)$$

The series representation of the $\bar{H}$-function is given by [6, p.271,Eqn.(6)]

$$\bar{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]$$

$$=\sum_{h=1}^{M}\sum_{r=0}^{\infty}\frac{\prod_{\substack{j=1\\j\neq h}}^{M}\Gamma(b_j-\beta_j s)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j s)\}^{A_j}(-1)^r z^s}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j s)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j s)\,r!\,\beta_h} \qquad \ldots(1.5)$$

where

$$s = \xi_{h,r} = \frac{b_h + r}{\beta_h}$$

The conventional formulation may fail pertaining to the domain of quantum cosmology but Feynman path integrals [7,8]. Feynman path integrals reformulation of quantum mechanics is more fundamental than the conventional formulation in terms of operators. Feynman integrals are useful in the study and development of simple and multiple variable hypergeometric series which in turn are useful in statistical mechanics.

2. MAIN RESULTS

Here we shall evaluate the following integrals pertaining to M-series [9] with $\overline{H}$ -function [7], a general class of polynomials [11].

(A)

$$\int_0^1 \int_0^1 \left[\frac{(1-x)}{(1-xy)}y\right]^{\tau} \left[\frac{1-y}{1-xy}\right]^{\psi} \left[\frac{(1-xy)}{(1-x)(1-y}\right] S_V^U\left[c_1\left\{\frac{(1-x)y}{(1-xy)}\right\}^{\lambda_1}\left\{\frac{(1-y)}{(1-xy)}\right\}^{\theta_1}\right]$$

$$.\ \overline{H}^{M,N}_{P,Q}\left[c_2\left\{\frac{(1-x)y}{(1-xy)}\right\}^{\lambda_2}\left\{\frac{(1-y)}{(1-xy)}\right\}^{\theta_2}\right] {}_{\alpha}\overset{\lambda}{M}{}_{\beta}\left[c_3\left\{\frac{(1-x)y}{(1-xy)}\right\}^{\lambda_3}\left\{\frac{(1-y)}{(1-xy)}\right\}^{\theta_3}\right]$$

$$.\ \overline{H}^{m,n}_{p,q}\left[c_4\left\{\frac{(1-x)y}{(1-xy)}\right\}^{\lambda_4}\left\{\frac{(1-y)}{(1-xy)}\right\}^{\theta_4}\right] dx\, dy$$

$$= \sum_{j=0}^{[V/U]} \sum_{h=1}^{M} \sum_{g,r=0}^{\infty} \frac{(-V)_{UJ}}{J\,!} A_{V,J} \frac{(-1)^r}{r!\,\beta_h} \phi(s)\, c_1^J c_2^s c_3^g \frac{(u_1)_g \ldots (u_\alpha)_g}{(v_1)_g \ldots (v_\beta)_g} \frac{1}{\Gamma(\lambda g + 1)}$$

$$.\ \overline{H}^{m,\,n+2}_{p+2,q+1}\left[c_4 \left| \begin{matrix} (1-\tau-\lambda_1 J-\lambda_2 s-\lambda_3 g,\lambda_4;1),(1-\psi-\theta_1 J-\theta_2 s-\theta_3 g,\theta_4;1),(e_j,\gamma_j;E_j)_{1,n},(e_j,\gamma_j)_{n+1,p} \\ (f_i,\delta_j)_{1,m},(f_j,\delta_j;F_j)_{m+1,q},\{1-\tau-(\lambda_1+\theta_1)J-(\lambda_2+\theta_2)s-(\lambda_3+\theta_3)g,(\lambda_4+\theta_4);1\} \end{matrix} \right.\right] \quad \ldots(2.1)$$

provided that

$$\mathrm{Re}\left(\tau+1+\lambda_2\frac{b_j}{\beta_j}+\lambda_4\frac{f_{j'}}{\delta_{j'}}\right)>0,\ \mathrm{Re}\left(\psi+1+\theta_2\frac{b_j}{\beta_j}+\theta_4\frac{f_{j'}}{\delta_{j'}}\right)>0,$$

$$|\arg c_2|<\frac{1}{2}\Omega'\pi,\Omega'>0,\ |\arg c_4|<\frac{1}{2}\Omega\pi,\Omega>0,$$

where $j=1,...,M;\ j'=1,...,m;\ (\tau,\psi,\lambda_1,\theta_1,\lambda_2,\theta_2,\lambda_3,\theta_3,\lambda_4,\theta_4)>0$ and

$$\alpha\le\beta,\ |c_3|<1,\ c_1>0.$$

(B)

$$\int_0^\infty\int_0^\infty f(x+y)\,x^{\tau-1}\,y^{\psi-1}\,S_V^U[c_1\,x^{\lambda_1}y^{\theta_1}]$$

$$.\ \overline{H}^{M,N}_{P,Q}\left[c_2x^{\lambda_2}y^{\theta_2}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]{}_{\alpha}\overset{\lambda}{M}{}_{\beta}[c_3\,x^{\lambda_3}y^{\theta_3}]$$

$$.\ \overline{H}^{m,n}_{p,q}\left[c_4\,x^{\lambda_4}y^{\theta_4}\left|\begin{matrix}(e_j,\gamma_j;E_j)_{1,n},(e_j,\gamma_j)_{n+1,p}\\(f_j,\delta_j)_{1,m},(f_j,\delta_j;F_j)_{m+1,q}\end{matrix}\right.\right]dx\,dy$$

$$=\sum_{j=0}^{[V/U]}\sum_{h=1}^{M}\sum_{g,r=0}^{\infty}\frac{(-V)_{UJ}}{J!}A_{V,J}\frac{(-1)^r}{r!}\frac{\phi(s)}{\beta_h}\frac{(u_1)_g...(u_\alpha)_g}{(v_1)_g...(v_\beta)_g}\frac{1}{\Gamma(\lambda g+1)}c_1^Jc_2^sc_3^g$$

$$.\int_0^\infty f(z)\,z^{[\tau+\psi+(\lambda_1+\theta_1)J+(\lambda_2+\theta_2)s+(\lambda_3+\theta_3)g-1}\,dz$$

$$\times\overline{H}^{m,\,n+2}_{p+2,q+1}\left[c_4z^{\lambda_4+\theta_4}\left|\begin{matrix}(1-\tau-\lambda_1J-\lambda_2s-\lambda_3g,\lambda_4;1),(1-\psi-\theta_1J-\theta_2s-\theta_3g,\theta_4;1),(e_j,\gamma_j;E_j)_{1,n},(e_j,\gamma_j)_{n+1,p}\\(f_i,\delta_j)_{1,m},(f_j,\delta_j;F_j)_{m+1,q},\{1-\tau-\psi-(\lambda_1+\theta_1)J-(\lambda_2+\theta_2)s-(\lambda_3+\theta_3)g,(\lambda_4+\theta_4);1\}\end{matrix}\right.\right]$$

…(2.2)

provided that

$$\operatorname{Re}\left(\tau+\lambda_2\frac{b_j}{\beta_j}+\lambda_4\frac{f_{j'}}{\delta_{j'}}\right)>0, \operatorname{Re}\left(\tau+\lambda_2\left(\frac{a_j-1}{\alpha_j}\right)+\lambda_4\left(\frac{E_{j'}(e_{j'}-1)}{\gamma_{j'}}\right)\right)<0$$

$$\operatorname{Re}\left(\psi+\theta_2\frac{b_j}{\beta_j}+\theta_4\frac{f_{j'}}{\delta_{j'}}\right)>0,$$

$$\operatorname{Re}\left(\psi+\theta_2\left(\frac{a_j-1}{\alpha_j}\right)+\theta_4\left(\frac{E_{j'}(e_{j'}-1)}{\gamma_{j'}}\right)\right)<0$$

$$|\arg c_2|<\frac{1}{2}\Omega'\pi, \Omega'>0,$$

$$|\arg c_4|<\frac{1}{2}\Omega\pi, \Omega>0,$$

where $j = 1,\ldots,M$; $j'=1,\ldots,m; (\tau,\psi,\lambda_1,\theta_1,\lambda_2,\theta_2,\lambda_3,\theta_3,\lambda_4,\theta_4)>0$ and

$$\alpha\le\beta, |c_2|<1, c_1>0.$$

(C)

$$\int_0^1\int_0^1 f(xy)(1-x)^{\tau-1}(1-y)^{\psi-1}y^{\tau}S_V^U[c_1(1-x)^{\lambda_1}(1-y)^{\theta_1}y^{\lambda_1}]$$

$$\overline{H}_{P,Q}^{M,N}\left[c_2(1-x)^{\lambda_2}(1-y)^{\theta_2}y^{\lambda_2}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]{}_{\alpha}\overset{\lambda}{M}{}_{\beta}[c_3(1-x)^{\lambda_3}(1-y)^{\theta_3}y^{\lambda_3}]$$

$$.\ \overline{H}_{p,q}^{m,n}\left[c_4(1-x)^{\lambda_4}(1-y)^{\theta_4}y^{\lambda_4}\left|\begin{matrix}(e_j,\gamma_j;E_j)_{1,n},(e_j,\gamma_j)_{n+1,p}\\(f_j,\delta_j)_{1,m},(f_j,\delta_j;F_j)_{m+1,q}\end{matrix}\right.\right]dx\,dy$$

$$=\sum_{j=0}^{[V/U]}\sum_{h=1}^{M}\sum_{g,r=0}^{\infty}\frac{(-V)_{UJ}}{J!}A_{V,J}\frac{(-1)^r}{r!}\frac{\phi(s)}{\beta_h}\frac{(u_1)_g\ldots(u_\alpha)_g}{(v_1)_g\ldots(v_\beta)_g}\frac{1}{\Gamma(\lambda g+1)}c_1^Jc_2^sc_3^g$$

$$.\int_0^1 f(z)(1-z)^{\tau+\psi+(\lambda_1+\theta_1)J+(\lambda_2+\theta_2)s+(\lambda_3+\theta_3)g-1}dz$$

$$\times \overline{H}^{m,n+2}_{p+2,q+1}\left[c_4(1-z)^{(\lambda_4+\theta_4)}\left|\begin{matrix}(1-\tau-\lambda_1 J-\lambda_2 s-\lambda_3 g,\lambda_4;1),(1-\psi-\theta_1 J-\theta_2 s-\theta_3 g,\theta_4;1),(e_j,\gamma_j;E_j)_{1,n},(e_j,\gamma_j)_{n+1,p}\\(f_i,\delta_j)_{1,m},(f_j,\delta_j;F_j)_{m+1,q},\{1-\tau-\psi-(\lambda_1+\theta_1)J-(\lambda_2+\theta_2)s-(\lambda_3+\theta_3)g,(\lambda_4+\theta_4);1\}\end{matrix}\right.\right]$$

…(2.3)

provided that

$$\mathrm{Re}\left(\tau+1+\lambda_2\frac{b_j}{\beta_j}+\lambda_4\frac{f_{j'}}{\delta_{j'}}\right)>0,\ \mathrm{Re}\left(\lambda+\lambda_2\frac{b_j}{\beta_j}+\lambda_4\frac{f_{j'}}{\delta_{j'}}\right)>0$$

$$\mathrm{Re}\left(\psi+\theta_2\frac{b_j}{\beta_j}+\theta_4\frac{f_{j'}}{\delta_{j'}}\right)>0,$$

$$|\arg c_2|<\frac{1}{2}\Omega'\pi,\ \Omega'>0,$$

$$|\arg c_4|<\frac{1}{2}\Omega\pi,\ \Omega>0,$$

where j = 1,…,M ; $j'=1,...,m;(\tau,\psi,\lambda_1,\theta_1,\lambda_2,\theta_2,\lambda_3,\theta_3,\lambda_4,\theta_4)>0$ and

$$\alpha\le\beta,\ |c_3|<1,\ c_1>0.$$

(D)

$$\int_0^1\int_0^1\left[\frac{(1-x)y}{1-xy}\right]^{\tau+\psi}\left[\frac{1-y}{1-xy}\right]^{\theta}\left[\frac{1}{(1-x)}\right]P_n(\tau,\theta)\left[1-2c_1\left\{\frac{y(1-x)}{1-xy}\right\}^{\lambda_1}\left\{\frac{1-y}{1-xy}\right\}^{\theta_1}\right]$$

$$.S_V^U\left[c_2\left\{\frac{y(1-x)}{(1-xy)}\right\}^{\lambda_2}\left\{\frac{1-y}{1-xy}\right\}^{\theta_2}\right]$$

$$\overline{H}^{M,N}_{P,Q}\left[c_3\left\{\frac{y(1-x)}{1-xy}\right\}^{\lambda_3}\left\{\frac{1-y}{1-xy}\right\}^{\theta_3}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]$$

$$\cdot {}_{\alpha}\overset{\lambda}{M}_{\beta}\left[c_4\left\{\frac{y(1-x)}{1-xy}\right\}^{\lambda_4}\left\{\frac{1-y}{1-xy}\right\}^{\theta_4}\right]$$

$$\cdot\ \overline{H}^{m,n}_{p,q}\left[c_5\left\{\frac{y(1-x)}{1-xy}\right\}^{\lambda_5}\left\{\frac{1-y}{1-xy}\right\}^{\theta_5}\left|\begin{matrix}(e_j,\gamma_j;E_j)_{1,n},(e_j,\gamma_j)_{n+1,p}\\(f_j,\delta_j)_{1,m},(f_j,\delta_j;F_j)_{m+1,q}\end{matrix}\right.\right]dx\,dy$$

$$=\sum_{j=0}^{[V/U]}\sum_{h=1}^{M}\sum_{g,r=0}^{\infty}\frac{(-V)_{UJ}}{J!}A_{V,J}\frac{(-1)^r}{r!\,\beta_h}\phi(s)\frac{(u_1)_g\ldots(u_\alpha)_g}{(v_1)_g\ldots(v_\beta)_g}\frac{1}{\Gamma(\lambda g+1)}c_2^J c_3^s c_4^g\frac{(1+\tau)_n}{n!}$$

$$\cdot\ {}_2F_1\left[\begin{matrix}-n,\tau+\theta+n+1;\\1+\tau\quad ;\end{matrix}c_1\right]$$

$$\cdot\ \overline{H}^{m,n+2}_{p+2,q+1}\left[c_5\left|\begin{matrix}(1-\tau-\psi-\lambda_1k_1-\lambda_2J-\lambda_3s-\lambda_4g,\lambda_5;1),(-\theta-\theta_1k_1-\theta_2J-\theta_3s,\theta_4g,\theta_5;1),(e_j,\gamma_j;E_j)_{1,n},(e_j,\gamma_j)_{n+1,p}\\(f_i,\delta_j)_{1,m},(f_j,\delta_j;F_j)_{m+1,q},\{-\tau-\theta-\psi-(\lambda_1+\theta_1)k_1-(\lambda_2+\theta_2)J-(\lambda_3+\theta_3)s-(\lambda_4+\theta_4)g,(\lambda_5+\theta_5);1\}\end{matrix}\right.\right]$$

$$\ldots(2.4)$$

provided that

$$\mathrm{Re}\left(\tau+\psi+1+\lambda_3\frac{b_j}{\beta_j}+\lambda_5\frac{f_{j'}}{\delta_{j'}}\right)>0,\ \mathrm{Re}\left(\theta+1+\theta_3\frac{b_j}{\beta_j}+\theta_5\frac{f_{j'}}{\delta_{j'}}\right)>0,$$

$$.\ |\arg c_3|<\frac{1}{2}\Omega'\pi,\Omega'>0,\ |\arg c_5|<\frac{1}{2}\Omega\pi,\Omega>0,$$

where $j=1,\ldots,M;\ j'=1,\ldots,m;\ (\tau,\psi,\theta,\lambda_1,\theta_1,\lambda_2,\theta_2,\lambda_3,\theta_3,\lambda_4,\theta_4,\lambda_5,\theta_5)>0$ and

$$\alpha\le\beta,\ |c_4|<1.$$

Proofs. The results (2.1) through (2.4) can be easily proved by using the same method as used by Chaurasia and Shekhawat [3].

3. PARTICULAR CASES

(3.A) By setting $A_j = B_j = 1, j = 1,...,M; j = 1,...,N)$ in (2.1) through (2.4), we get the known results obtained by Chaurasia and Jagdev Singh ([2], eqn. (2.1), p.2,eqn.(2.2), p.2, eqn.(2.3), p.3, eqn. (2.4), p.4).

(3.B) By taking $\lambda = 1$, for the generalized hypergeometric function [9] and $A_j = B_j = 1$, $j = 1,\ldots,M$; $j = 1,\ldots,N$ in the results (2.1) through (2.4), we get the known results obtained by Chaurasia and Jagdev Singh ([2],eqn. (A.1) through (A.2), p.4,5).

(3.C) Using $A_j = B_j = 1, \lambda_1 \to 1, \theta_1 \to 0, \lambda_2 \to 0, \theta_2 \to 0, c_2 \to 1, \lambda_3 \to 0, \theta_3 \to 0,$

$c_4 \to 1$ in equation (2.1), $A_j = B_j = 1, c_1 \to 0, \lambda_1 \to 1, \theta_1 \to 0, c_2 \to 1, \lambda_2 \to 0, \theta_2 \to 0,$

$\lambda_3 \to 0$, in equation (2.2) and (2.3), we obtain the known results established by Chaurasia and Shekhawat ([3], eqn. 2.1`, p.184, eqn. (2.3), p.185, and eqn. (2.5),p.186).

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