LINEAR MAPS ON METAPHORS OF INFINITE DIRECT CREATE ALGEBRAS

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ABSTRACT.

Let k be an infinite field, I an infinite set, V be a k-Vector-space, and $g : k^I \to V$ a k-linear map. It is shown that if $\dim_k(V)$ is not too large (under various hypotheses on card(k)and card(I), if it is finite, respectively less than card(k), respectively less than the continuum), then $k$er(g)must contain elements $(u_i)_{i\in I}$ with all but finitely many components u_i nonzero. These results are used to prove that every homomorphism from a direct product $\Pi_I A_i$ of not-necessarily associative algebras A_i onto an algebra B, where $\dim_k(B)$ is not too large (in the same senses) is the sum of a map factoring through the projection $\Pi_I A_i$ onto the product of finitely many of the A_i, and a map into the ideal $\{b \in B \mid bB = Bb = \{0\}\} \subseteq B$.Detailed consequences are noted in the case where the A_i are Lie algebras. A version of the above result is also obtained with the field k replaced by a commutative valuation ring. This note resembles in that the two papers obtain similar results on homomorphisms on infinite product algebras; but the methods are different, and the hypotheses under which the methods of one note work are in some ways stronger, in others weaker, than those of the other. Also, in we obtain many consequences from our results, while here we aim for brevity, and after one main result about general algebras, restrict ourselves to a couple of quick consequences for Lie algebras. .

Key words:Resembles, Measurable cardinals ,Nilpotent Lie algebras,

1. INTRODUCTION AND DEFINITIONS

Let us fix some terminology and notation.

Definition 1.

Throughout this note, k will be a field.

By an algebra over k we shall mean a k-vector-space A given with a k-bilinear multiplication $A \times A \to A$, which we do not assume associative or unital.

If A is an algebra, we define its total annihilator ideal to be

(1) $Z(A) = \{x \in A \mid xA = Ax = \{0\}\}$.

If $a = (a_i)_{i\in I}$ is an element of a direct product algebra $A = \Pi_I A_i$, then we define its support

(2) $\mathrm{supp}(a) = \{i \in I \mid a_i \neq 0\}$.

For J any subset of I, we shall identify $\Pi_{i\in J} A_i$ with the subalgebra of $\Pi_{i\in I} A_i$ consisting of elements whose support is contained in J. We also define the subalgebra

(3) $A_{fin} = \{a \in A \mid \mathrm{supp}(a) \text{ is finite}\}$.

(4) $g_a : k^I \to B$ defined by $g_a((u_i)) = f((u_i a_i))$ for all $(u_i) \in k^I$.

Lemma 1

(i) If $\ker(g_a)$ contains an element $u = (u_i)_{i\in I}$ whose support is all of I, then $f(a) \in Z(B)$.

(ii) More generally, for any $u \in \ker(g_a)$, if we write $a = a' + a''$, where $\mathrm{supp}(a') \subseteq \mathrm{supp}(u)$ and $\mathrm{supp}(a'') \subseteq I - \mathrm{supp}(u)$, then $f(a') \in Z(B)$.

(iii) Hence, if $\ker(g_a)$ contains an element whose support is cofinite in I, then a is the sum of an element $a' \in f^{-1}(Z(B))$ and an element $a'' \in A_{fin}$.

Proof. (i): Given u as in (i), and any $b \in B$, let us write $b = f(x)$, where $x = (x_i) \in A$, and compute

(5) $f(a)\, b = f(a) f(x) = f(a\, x) = f((a_i x_i)) = f((u_i a_i u_i^{-1} x_i))$

$= f((u_i a_i))\, f((u_i^{-1} x_i)) = 0\, f((u_i^{-1} x_i)) = 0.$

So f (a) left-annihilates all elements of B; and by the same argument with the order of factors reversed,

ISSN 2278-7763

itright-annihilates all elements of B. Thus, $f(a) \in Z(B)$, as claimed.

(ii): Let $u' \in k^I$ be defined by taking $u_i' = u_i$ for $i \in supp(u)$, and $u_i' = 1$ for $i \notin supp(u)$. Thus, $supp(u') = I$; moreover, $u'a' = ua$, whence $f(u'a') = f(ua) = 0$. Hence, $ker(g_{a'})$ contains the element u whose support is I; so by (i), $f(a') \in Z(B)$.

(iii) clearly follows from (ii).

Motivated by statement (iii) of the lemma, let us look for conditions under which thekernel of a homomorphism on k^I must contain elements of co-finite support. Here is an easy one.

Lemma 2.

Let I be a set with $card(I) \leq card(k)$, and $g : k^I \to V$ ak-linear map, for some finite dimensional k-vector-space V. Then there exists $u \in ker(g)$ such that $I - supp(u)$ is finite.

Proof. By the assumption on card(I), we can choose $x = (x_i) \in k^I$ whose entries x_i are distinct. Regarding k^I as a k-algebra under component wise operations, let us map the polynomial algebra $k[t]$ into it by the homomorphism sending t to this x. Composing with $g : k^I \to V$, we get ak-linear map $k[t] \to V$.

Since V is finite-dimensional, this map has nonzero kernel, so we may choose $0 = p(t) \in k[t]$ such that $p(x) \in ker(g)$. Since the polynomial p has only finitely many roots, $p(x_i)$ is zero for only finitely many i, so $p(x)$ gives the desired u.

Applying Lemma 3 to maps g_a as in Lemmas 2, and calling on statement (iii) of the latter, we get

Lemma 3. Let k be an infinite field, let $(A_i)_{i \in I}$ be a family ofk-algebras such that the index set has cardinality $\leq card(k)$, let $A = \ \ I\, A_i$, and let $f : A \to B$ be any surjective algebra homomorphism to a finite-dimensional k-algebra B.

Then $B = f(A_{fin}) + Z(B)$. (Equivalently, $A = A_{fin} + f^{-1}(Z(B))$.)

Hence B is the sum of Z(B) and the (mutually annihilating) images of finitely many of the A_i.

Proof. The first assertion follows immediately from the two preceding lemmas. To get the final assertion, we note that since B is finite-dimensional, its subalgebra $f(A_{fin}) = f(\oplus_I A_i) = \Sigma_I f(A_i)$ must be spanned by the images of finitely many of the A_i, and since the A_i, as subalgebras of A, annihilate one another, so do those images.

In the next two sections we shall obtain three strengthenings of Lemma 3, two of which weaken the assumption of finite-dimensionality of V, while the third, instead, weakens the restriction on card(I)

Our first generalization of Lemma 3 will be obtained by replacing the countable-dimensional polynomial ring $k[t]$ by a subspace of the rational function field $k(t)$ which has dimension card(k) overk. Rational functions are not, strictly speaking, functions; but that will be easy to fudge.

Lemma 4. For each $c \in k$, let $p^{(c)} \in k^k$ be the function which for every $x \in k - \{c\}$ has $p^{(c)}(x) = (x-c)^{-1}$, and at c has the value 0. Then any nontrivial linear combination of the elements p(c) has at most finitely many zeroes.

Hence if I is a set of cardinality $\leq card(k)$, and g is ak-linear map of k^I to ak-vector-space V of dimension $< card(k)$, then ker(g) contains an element u of cofinite support.

Proof. In k(t), any linear combination of elements $(t - c_1)^{-1}, \ldots, (t - c_n)^{-1}$ for distinct $c_1, \ldots, c_n \in k$ $(n \geq 1)$, such that each of these elements has nonzero coefficient, gives a nonzero rational function

(6) $a_1(t - c_1)^{-1} + \cdots + a_n(t - c_n)^{-1} = h(t)/((t - c_1)\ldots(t - c_n))$ (where $h(t) \in k[t]$).

Indeed, to see that (6) is nonzero in k(t), multiply by any $t - c_m$. Then we can evaluate both sides at c_m, and we find that the left-hand side then has a unique nonzero term; so we must have $h(c_m) \neq 0$. Hence $h(t)$ is a nonzero element of k[t], so (6) is a nonzero element of k(t).

If we now take the corresponding linear combination of $p^{(c1)}, \ldots, p^{(cn)}$ in k^k, the result has the value $h(c)/((c - c_1) \ldots (c - c_n))$ at each $c \neq c_1, \ldots, c_n$. Hence it is nonzero everywhere except at the finitely many zeroes of $h(t)$, and some subset of the finite set $\{c_1, \ldots, c_n\}$.

We get the final assertion by embedding the set I in k, so that the $p^{(c)}(c \in k)$ induce elements of k^I. These will form a card(k)-tuple of functions, any nontrivial linear combination of which is a function with only finitely many zeroes. Under a linear map g from k^I to a vector space V of dimension < card(k), some nontrivial linear combination u of these card(k) elements must go to zero, yielding a member of ker(g) with the asserted property.

(An alternative way to get around the problem that rational functions have poles would be to partition k into two disjoint subsets of equal cardinalities, and use linear combinations of rational functions $1/(t - c)$ with c ranging over one of these sets to get functions on the other.)

For k countable, the condition on the dimension of V in the final statement of the above lemma is no improvement on what we got in Lemma 3 using k[t] In an earlier version of this note, we obtained an improvement on Lemma 3 for countable k by a diagonal argument, showing that if k and I are both countably infinite, then any maximal subspace $W \subseteq k^I$ no nonzero member of which has infinitely many zero coordinates must be uncountable-dimensional. Jason Bell communicated to us the following stronger result, which not only gives a subspace of continuum, rather than merely uncountable, dimension, but (as is made clear in the proof, though for simplicity we do not include it in the statement), also shares with the constructions of Lemmas 3 and 5 the property that for every finite-dimensional subspace of W, there is a uniform bound on the number of zero coordinates of its nonzero elements, which our earlier result lacked. (The result below was, in fact, given in response to the question we raised of whether a construction admitting such uniform bounds was possible.)

Lemma 5(sketched by Jason Bell, personal communication). If the field k is infinite, and I is a countably infinite set, then there exists a subspace $W \subseteq k^I$ of continuum dimensionality such that no nonzero member of W has infinitely many zeroes.

Hence any k-linear map g from k^I to a k-vector-space V of less than continuum dimension has in its kernel an element u of cofinite support.

Proof. It suffices to prove the stated result for $I = \omega$, the set of natural numbers.

Let us first note that if k is either of characteristic 0, or transcendental over its prime field, then it is algebraic over a Unique Factorization Domain R which is not a field (namely, Z, or a polynomial ring over the prime field of k). This ring R admits a discrete valuation, which induces a discrete valuation on the field of fractions of R. It is easily deduced from [12, Prop. XII.4.2] that this extends to a Q-valued valuation v on the algebraic extension k of that field, and by rescaling, v can be assumed to have valuation group containing Z. Let us call this situation Case I.

If we are not in Case I, then k must be an infinite algebraic extension of a finite field. Hence it will contain a countable chain of distinct subfields,

(7) $k_0 \subset k_1 \subset \cdots \subset k_i \subset \cdots$.

Given any field k containing such a chain of subfields (regardless of characteristic, or algebraicity over a prime field), we may define a natural-number-valued function v (not a valuation) on $\bigcup_{i \in \omega} k_i \subseteq k$ by letting v(x) be the least i such that $x \in k_i$. We shall call the situation where k contains a chain (7) Case II. (So Cases I and II together cover all infinite fields, with a great deal of overlap.)

In either case, let us choose elements $x_0, x_1, \cdots \in k$ such that

(8) $v(x_i) = i$ for all $i \in \omega$, and for every real number $\alpha > 1$, let $f_\alpha \in k^\omega$ be defined by

(9) $f_\alpha(n) = x_{[\alpha n]}$ $(n \in \omega)$,

where $[\alpha n]$ denotes the largest integer $\leq \alpha n$.

This gives continuum many elements $f_\alpha \in k\omega$. We shall now complete the proof by showing separately in Cases I and II that for any $1 < \alpha_1 < \cdots < \alpha_d$, there exists a natural number N such that no nontrivial linear combination

(10) $c_1 f_{\alpha_1} + \cdots + c_d f_{\alpha_d} c_1 (c_1, \ldots, c_d \in k)$

has more than N zero coordinates.

If we are in Case I, consider any n such that the n-th coordinate of (10) is zero. This says that

(11) $\sum_i c_i x_{[n\alpha_i]} = 0$

Now if a family of elements ofkwhich are not all zero has zero sum, then at least two nonzero members ofthe family must have equal valuation. Thus, for some i< j with $c_i, c_j \neq 0$ we have

(12) $v(c_i) + v(x_{[n\alpha_i]}) = v(c_j) + v\left(x_{[n\alpha_j]}\right)$

By (8), this says

(13) $v(c_i) + \lfloor n\alpha_i \rfloor = v(c_j) + \lfloor n\alpha_j \rfloor$

From the fact that $[n\alpha_i]$ lies in the interval $(n\alpha_i - 1, n\alpha_i]$, and the corresponding fact for $n\alpha_j$, we seethat $[n\alpha_i] - [n\alpha_j]$ differs by less than 1 from $n\alpha_i - n\alpha_j$, so (13) implies

(14) $n(\alpha_j - \alpha_i) \in (v(c_i) - v(c_j) - 1,\ v(c_i) - v(c_j) + 1)$.

This puts n in an open interval of length $2/(\alpha_j - \alpha_i)$. We have shown that whenever the n-th coordinateof (10) is zero, this relation holds for some pair i, j; so the total number of possibilities for n is at most

(15) $N = \Sigma_{i<j}[2/(\alpha_j - \alpha_i)]$,

a bound depending only on $\alpha_1, \ldots, \alpha_d$(and not on $c_1, \ldots, c_d$), as claimed.

Next, suppose we are in Case II. Then we claim that for an element (10), there can be at most d – 1values of n with

(16) $n \geq \max_{i=1,\ldots,d-1}(1/(\alpha_{i+1} - \alpha_i))$

for which the n-th coordinate of (10) is zero. For suppose, on the contrary, that $n_1 < \cdots < n_d$ all have this property. This says that the nonzero column vector of coefficients $(c_1, \ldots, c_d)^T$ is left annihilated by the d × d matrix

(17) $\left(\left(x_{[n_i\alpha_j]}\right)\right)$

Note that the subscripts $n_i\alpha_j$ in (17) are strictly increasing in both i and j; the former because all $\alpha_j > 1$, the latter because all n_i satisfy (16). It follows that in the matrix (17), every minor has the property that its lower right-hand entry does not lie in the subfield generated by its other entries. From this, it is easy to show by induction that all minors have nonzero determinant, and so in particular that (17) is invertible.

But this contradicts the assumption that (17) annihilates $(c_1, \ldots, c_d)^T$. Hence there are, as claimed, at most d − 1 values of n satisfying (16) such that the n-th entry of (10) is zero; so the total number of zero entries of (10) is bounded by

(18) $N = \max_{i=1,\ldots,d-1} [1/(\alpha_{i+1} - \alpha_i)] + d$,

which again depends only on the α_i.

The final assertion of the lemma clearly follows.

Remark: In Case I of the above proof, in place of condition (8) we could equally well have used x_i with $v(x_i) = -i$. Similarly, the proof in Case II can be adapted to fields k having a descending chain of subfields $k = k_0 \supset k_1 \supset \cdots \supset k_i \supset \cdots$: in this situation, we define v on $k - \cap_{i \in \omega} k_i$ to take each x to the largest i such that $x \in k_i$, and consider upper left-hand corners of minors instead of lower right-hand corners. We know of no use for these observations at present; but they might be of value in proving some variants of the above lemma.

For our third generalization of Lemma 2, we return to the hypothesis that v is finite-dimensional, and prove that in that situation, the statement that every linear map $g : k^I \to v$ has elements of cofinite support in fact holds for sets I of cardinality much greater than card(k).

We can no longer get this conclusion by finding an infinite-dimensional subspace $W \subseteq k^I$ whose nonzero members each have only finitely many zeroes.

On the contrary, when card(I) > card(k) (with the former infinite) there can be no subspace $W \subseteq k^I$ of dimension > 1 whose nonzero members all have only finitely many zeroes. For if (x_i) and (y_i) are linearly independent elements of W, and we look at the subspacesofk^2generated by the pairs (x_i, y_i) as i runs over I, then if card(I) > card(k), at least one of these
subspaces must occur at card(I) many values of i, but cannot occur at all i; hence some linear combination of (x_i) and (y_i) will have card(I) zeroes, but not itself be zero. So we must construct our elements of cofinite support in a different way, paying attention to the particular map g.

Surprisingly, our proof will again use the polynomial trick of Lemma 3; though this time only after considerable preparation. (We could use rational functions in place of these polynomials as in Lemma 5,or functions like the fα of Lemma 6, but so far as we can see, this would not improve our result, since finite-dimensionality of v is required by other parts of the argument.)

The case of Theorem 9 below that we will deduce from the result of this section is actually slightly weaker than the corresponding result proved by different methods in [3]. Hence the reader who is only interested in consequences for algebra homomorphisms $\Pi_I\ A_i \to B$ may prefer to skip the lengthy and intricate argument of this section. On the other hand, insofar as our general technique makes the question, "For what k, I and V can we say that the kernel of every k-linear map $k^I \to v$ must contain an element of cofinite support?"
itself of interest, the result of this section creates a powerful complement to those of the preceding section.
We will assume here familiarity with the definitions of ultrafilter and ultraproduct (given in most books on universal algebra or model theory, and summarized in [3, §14]), and of κ-completeness of an ultrafilter (developed, for example, in [7] or [8], and briefly summarized in the part of [3, §15] preceding Theorem 47).In the lemma below, we do not yet restrict card(I) at all. As a result, we will get functions with zero-sets characterized in terms of finitely many $card(k)^+$-complete ultrafilters, rather than finitely many points. Inthe corollary to the lemma, we add a cardinality restriction which forces such ultrafilters to be principal, and so get elements with only finitely many zeroes. The lemma also allows k to be finite, necessitating aproviso (19) that its cardinality not be too small relative to $\dim_k(V)$; this, too, will go away in the corollary,
where, for other reasons, we will have to require k to be infinite.

In reading the lemma and its proof, the reader might bear in mind that the property (21) makes J_0"good" for our purposes, while $J_1, \ldots, J_n$ embody the complications that we must overcome. The case of property (21) that we will want in the end is for the element $0 \in g(k^{J_0})$; but in the course of the proof it will be important to consider that property for arbitrary elements of that subspace.

Lemma 6.Let I be a set, V a finite-dimensional k-vector space such that

(19) $\mathrm{card}(k) \geq \dim_k(V) + 2$,

and $g : k^I \to V$ ak-linear map.

Then I may be decomposed into finitely many disjoint subsets,

(20) $I = J_0 \cup J_1 \cup \ldots \cup J_n$

($n \geq 0$), such that

(21) every element of $g(k^{J_0})$ is the image under g of an element having support precisely J_0, and such that each set J_m for $m = 1, \ldots, n$ has on it a $card(k)^+$-complete ultrafilter U_m such that, letting ψ denote the factor-map $V \to V/g(k^{J_0})$, the composite $\psi_g : k^I \to V/g(k^{J_0})$ can be factored

(22) $k^I = k^{J_0} \times k^{J_1} \times \cdots \times k^{J_n} \to k^{J_1}/U_1 \times \cdots \times k^{J_n}/U_n \to V/g(k^{J_0})$,

Where k^{J_m}/U_m denotes the ultrapower of k with respect to the ultrafilterU_m, the first arrow of (22) is theproduct

ISSN 2278-7763

of the natural projections, and the last arrow is an embedding.

Proof. If card(k) = 2, then (19) makes V = {0}, and the lemma is trivially true (with J_0= I and n = 0);so below we may assume card(k) > 2.

There exist subsets $J_0 \subseteq I$ satisfying (21); for instance, the empty subset. Since V is finite-dimensional,we may choose a J_0satisfying (21) such that

(23) Among subsets of I satisfying (21), J_0maximizes the subspace $g(k^{J0}) \subseteq V$,

i.e., such that no subset J_0' satisfying (21) has $g(k^{J'0})$ properly larger than $g(k^{J0})$.

Given this J_0,we now consider subsets $J \subseteq I - J_0$such that

(24)$g(k^J) \not\subseteq g(k^{J0})$, and J minimizes the subspace $g(k^{J0}) + g(k^J)$ subject to this

condition, in the sense that every subset $J' \subseteq J$ satisfies either

(25) $g(k^{J'}) \subseteq g(k^{J0})$

or

(26) $g(k^{J0}) + g(k^J) = g(k^{J0}) + g(k^J)$.

It is not hard to see from the finite-dimensionality of V, and the fact that inclusions of sets J imply the corresponding inclusions among the subspaces $g(k^{J0}) + g(k^J)$, that such minimizing subsets J will exist if $g(k^{J0}) \neq g(k^I)$. If, rather, $g(k^{J0}) = g(k^I)$, then the collection of such subsets that we develop in the arguments below will be empty, but that will not be a problem.

Let us, for the next few paragraphs, fix such a J. Thus, every $J' \subseteq J$ satisfies either (25) or (26).However, we claim that there cannot be many pair wise disjoint subsets $J' \subseteq J$ satisfying (26). Precisely, letting

(27) $e = \dim_k((g(k^{J0}) + g(k^J))/g(k^{J0}))$,

we claim that there cannot be 2e such pair wise disjoint subsets.

For suppose we had pair wise disjoint sets $J'_{\alpha,d} \subseteq J$ ($\alpha \in \{0, 1\}$, $d \in \{1, \ldots, e\}$) each satisfying (26). Let

(28) $h_1, \ldots, h_e \in g(k^{J0}) + g(k^J)$

be a minimal family spanning $g(k^{J0}) + g(k^J)$ over $g(k^{J0})$. For each $\alpha \in \{0, 1\}$ and $d \in \{1, \ldots, e\}$, condition (26) on $J'_{\alpha,d}$ allows us to choose an element $x^{(\alpha,d)} \in k^{J'\alpha,d}$ such that

(29) $g(x^{(\alpha,d)}) \equiv h_d (\mathrm{mod}\ g(k^{J0}))$.

Some of the $x^{(\alpha,d)}$ may have support strictly smaller than the corresponding set $J'_{\alpha,d}$; if this happens, let us cure it by replacing $J'_{\alpha,d}$ by $\mathrm{supp}(x^{(\alpha,d)})$: these are still pair wise disjoint subsets of J, and will stillsatisfy (26) rather than (25), since after this modification, the subspace $g(k^{J'\alpha,d})$ still contains $g(x^{(\alpha,d)}) \notin g(k^{J0})$.

We now claim that the set

(30) $J_0^* = J_0 \cup \bigcup_{\alpha \in \{0,1\} \text{ and } d \in \{1,\ldots,e\}} J'_{\alpha,d}$

contradicts the maximality condition (23) on J_0. Clearly $g(k^{J_0^*}) = g(k^{J_0}) + g(k^J)$ is strictly larger than $g(k^{J_0})$. To show that J_0^* satisfies the analog of (21), consider any $h \in g(J_0^*) = g(k^{J_0}) + g(k^J)$, and let us write it, using the relative spanning set (28), as

(31)

$$h = h_0 + c_1 h_1 + \cdots + c_e h_e \quad (h_0 \in g(k^{J_0}), c_1, \ldots, c_e \in k).$$

Since card(k) > 2, we can now choose for each $d = 1, \ldots, e$ an element $c'_d \in k$ which is neither 0 nor c_d

and form the element

(32) $$x = \left(c'_1 x^{(0,1)} + (c_1 - c'_1) x^{(1,1)} + c'_2 x^{(0,2)} + (c_2 - c'_2) x^{(1,2)} + \cdots + c'_e x^{(0,e)} + (c_e - c'_e) x^{(1,e)}\right).$$

By our choice of $c'_1, \ldots, c'_e$, none of the coefficients c'_d or $c_d - c'_d$ is zero, so $\mathrm{supp}(x) \cup J'_{\alpha,d}$. Applyingg to (32), we see from (29) that g(x) is congruent modulo $g(k^{J_0})$ to $c_1 h_1 + \cdots + c_e h_e$, hence, by (31),congruent to h. By (21), we can find an element $y \in k^{J_0}$ with support precisely J_0 that makes up the difference, so that $g(y) + g(x) = h$. The element $y + x$ has support exactly J_0^*; and since we have obtained an arbitrary $h \in g(J_0^*)$ as the image under g of this element, we have shown that J_0^* satisfies the analog of (21), giving the desired contradiction.

Thus, we have a finite upper bound (namely, $2e-1$) on the number of pairwise disjoint subsets J'that J can contain which satisfy (26). So starting with J, let us, if it is the union of two disjoint subsets with that property, split one off and rename the other J, and repeat this process as many times as we can. Then in finitely many steps, we must get a J which cannot be further decomposed. Summarizing what we know about this J, we have

(33)$g(k^J) \nsubseteq g(k^{J_0})$, every subset $J' \subseteq J$ satisfies either$g(k^{J'}) \subseteq g(k^{J_0})$ or $g(k^{J_0}) + g(k^{J'}) = g(k^{J_0}) + g(k^J)$, and no two disjoint subsets of J satisfy the latter equality.

Let us call any subset $J \subseteq I - J_0$ satisfying (33) a nugget. From the above development, we see that
(34) Every subset $J \subseteq I - J_0$such that $g(k^J) \nsubseteq g(k^{J_0})$, contains a nugget.
The rest of this proof will analyze the properties of an arbitrary nugget J, and finally show (after apossible adjustment of J_0 that $I - J_0$ can be decomposed into finitely many nuggets $J_1 \cup \cdots \cup J_n$, andthat these will have the properties in the statement of the proposition.
We begin by showing that
(35) If J is a nugget, then the set $\mathcal{U} = \{J' \subseteq J \mid g(k^{J_0}) + g(k^{J'}) = g(k^{J_0}) + g(k^J)\}$is an ultrafilter on J.
To see this, note that by (33), the complement of μ within the set of subsets of J is also the set ofcomplements relative to J of members of $\mathcal{U}$, and is, furthermore, the set of all $J' \subseteq J$ such that $g(k^{J'}) \subseteq g(k^{J_0})$. The latter set is clearly closed under unions and passing to smaller subsets, hence $\mathcal{U}$, inversely, is closed under intersections and passing to larger subsets of J; i.e., $\mathcal{U}$ is a filter. Since $\emptyset \notin \mathcal{U}$, while the complement of any subset of J not in $\mathcal{U}$does belong to $\mathcal{U}$, $\mathcal{U}$ is an ultrafilter.

Let us show next that any nugget J has properties that come perilously close to making $J_0 \cup J$ acounterexample to the maximality condition (23) on J_0. By assumption, $g(k^{J_0 \cup J})$ is strictly larger than $g(k^{J_0})$. Now consider any $h \in g(k^{J_0 \cup J})$. We may write

(36) $h = g(w) + g(x)$,Where $w \in k^{J_0}$, $x \in k^J$

Suppose first that

(37) $h \notin g(k^{J_0})$

From (36) and (37) we see that$g(x) \notin g(k^{J_0})$, so $supp(x) \in \mathcal{U}$. Now take any element $x' \in k^J$which agreeswith x on supp(x), and has (arbitrary) nonzero values on all points of $J - supp(x)$. The element by whichwe have modified x to get xhas support in $J - supp(x)$, which is $\in \mathcal{U}$ / because $supp(x) \in \mathcal{U}$; hence$g(x') \equiv g(x) \pmod{g(k^{J_0})}$, hence by (36), $g(x') \equiv h \pmod{g(k^{J_0})}$. Hence by (21), we can find $z \in k^{J_0}$with support exactlyJ_0such that $g(z) + g(x) = h$. Thus, $z + x$is an element with support $J_0 \cup J$ whoseimage under gis h.

This is just what would be needed to make $J_0 \cup J$ satisfy (21), if we had proved it for all $h \in g(k^{J_0 \cup J})$;but we have only proved it for h satisfying (37) (which we needed to argue that supp(x) belonged to $\mathcal{U}$).We now claim that if there were any $x \in k^J$ with $supp(x) \in \mathcal{U}$satisfying$g(x) \in g(k^{J_0})$, then wewould be able to complete our argument contradicting (23). For modifying such an x by any element with complementary support in J, we would get an element with support exactly J whose image under g would still lie in $g(k^{J_0})$. Adding to this element the images under g of all elements of k^{J_0} with support equal to J_0, we would get images under g of certain elements with support exactly $J_0 \cup J$. Moreover, since J_0 satisfies (21), these sums would comprise all $h \in g(k^{J_0})$,i.e., just those

ISSN 2278-7763

values that were excluded by (37).In view of the resulting contradiction to (23), we have proved

(38) If J is a nugget, then every $x \in k^J$ with $\mathrm{supp}(x) \in \mathcal{U}$ satisfies $g(x) \notin g(k^{J_0})$.

We shall now use the "polynomial functions" trick to show that (38) can only hold if the ultrafilter $\mathcal{U}$ is $\mathrm{card}(k)^+$-complete. If k is finite, $\mathrm{card}(k)^+$-completeness is vacuous, so assume for the remainder of this paragraph that k is infinite. If $\mathcal{U}$ is not $\mathrm{card}(k)^+$-complete, we can find pair wise disjoint subsets $J_c \subseteq J(c \in k)$ with $J_c \notin \mathcal{U}$, whose union is all of J. Given these subsets, let $z \in k^J$be the element having, for each $c \in k$, the value $z_i = c$ at all $i \in J_c$. Taking powers of z under componentwise multiplication,we get elements $1, z, \ldots, Z^n, \ldots \in k^J$. Since V is finite-dimensional, some nontrivial linear combination$p(z)$ of these must be in the kernel of g. But as a nonzero polynomial, p has only finitely many roots ink, say $c_1, \ldots, c_r$. Thus $\mathrm{supp}(p(z)) = J - (J_{c_1} \cup \cdots \cup J_{c_r})$. Since $J \in \mathcal{U}$ and $J_{c_1}, \ldots, J_{c_r} \notin$, we get $\mathrm{supp}(p(z)) \in u$; but since $p(z) \in \ker(g)$, we have $g(p(z)) = 0 \in g(k^{J_0})$, contradicting (38). Hence

(39) For every nugget J, the ultrafilter of (35) is $\mathrm{card}(k)^+$-complete.

We claim next that (39) implies that for any nugget J,

(40) $\dim_k((g(k^{J_0})+g(k^J))/g(k^{J_0})) = 1$.

Indeed, fix $x \in k^J$ with support J, and consider any $y \in k^J$. If we classify the elements $i \in J$ according to the value of $y_i/x_i \in k$, this gives card(k) sets, so by $\mathrm{card}(k)^+$-completeness, one of them, say $\{i \mid y_i = c\, x_i\}$ (for some $c \in k$) lies in u. Hence $y - c\,x$ has support $\notin u$, so $g(y - c\,x) \in g(k^{J_0})$, i.e., modulo $g(k^{J_0})$, the element $g(y)$ is a scalar multiple of $g(x)$. So $g(x)$ spans $g(k^{J_0}) + g(k^J)$ modulo $g(k^{J_0})$.

Let us now choose for each nugget J an element x_J with support J. Thus, by the above observations,$g(x_J)$ spans $g(k^{J_0}) + g(k^J)$ modulo $g(k^{J_0})$. We claim that

(41) For any disjoint nuggets $J_1, \ldots, J_n$, the elements $g(x_{J_1}), \ldots, g(x_{J_n}) \in V$ are linearly

independent modulo $g(k^{J_0})$.

For suppose, by way of contradiction, that we had some relation

(42)$\sum_{m=1}^{n} c_m g(x_{J_m}) \in g(k^{J_0})$, with not all $c_m$zero.

If $n > \dim_k(V)$, then there must be a linear relation in V among $\leq \dim_k(V) + 1$ of the $g(x_{J_m}) \in V$, so in that situation we may (in working toward our contradiction) replace the set of nuggets assumed to satisfy a relation (42) by a subset also satisfying

(43) $n \leq \dim_k(V) + 1$,

and (42) by a relation which they satisfy. Also, by dropping from our list of nuggets in (42) any J_m such that $c_m = 0$, we may assume those coefficients all nonzero.

We now invoke for the third (and last) time the maximality assumption (23), arguing that in the above situation, $J_0 \cup J_1 \cup \cdots \cup J_n$would be a counterexample to that maximality.

For consider any

(44) $v \in g(k^{J_0 \cup J_1 \cup \cdots \cup J_n})$.

By (40) and our choice of $x_{J_1}, \ldots, x_{J_n}$, v can be written as the sum of an element of $g(k^{J_0})$ and an element $\sum d_m\, g(x_{J_m})$ with $d_1, \ldots, d_n \in k$. By (19) and (43), $\mathrm{card}(k) \geq \dim_k(V) + 2 > n$, hence we can choose an element $c \in k$ distinct from each of $d_1/c_1, \ldots, d_n/c_n$ (for the c_m of (42)), i.e., such that $d_m - c\,c_m \neq 0$for $m = 1, \ldots, n$. Thus, $\sum(d_m - c\,c_m) x_{J_m}$, which by

(42) has the same image in $v/g(k^{J_0})$ as our given element v, is a linear combination of $x_{J_1},\ldots,x_{J_n}$ with nonzero coefficients, hence has support exactly $J_1 \cup \cdots \cup J_n$. As before, we can now use (21) to adjust this by an element with support exactly J_0 so thatthe image under g of the resulting element x is v. Since x has support exactly $J_0 \cup J_1 \cup \cdots \cup J_n$, we have the desired contradiction to (23).

It follows from (41) that there cannot be more than $\dim_k(V)$ disjoint nuggets; so a maximal family of pairwise disjoint nuggets will be finite. Let $J_1,\ldots,J_n$ be such a maximal family.

In view of (34), the set $J = I - (J_0 \cup J_1 \cup \cdots \cup J_n)$ must satisfy $g(k^J) \subseteq g(k^{J_0})$, hence we can enlarge J_0 by adjoining to it that set J, without changing $g(k^{J_0})$, and hence without losing (21). We then have (20).

For $m = 1,\ldots,n$, let u_m be the ultrafilter on J_m described in (35). To verify the final statement of the proposition, that there exists a factorization (22), note that any element of k^I can be written $a^{(0)} + a^{(1)} + \cdots + a^{(n)}$ with $a^{(m)} \in k^{J_m} (m = 0,\ldots,n)$, hence its image under g will be congruent modulo $g(k^{J_0})$ to $g(a^{(1)}) + \cdots + g(a^{(n)})$. Now the image of each $g(a^{(m)})$ modulo $g(k^{J_0})$ is a function only of theequivalence class of $a^{(m)}$ with respect to the ultrafilter u_m (since two elements in the same equivalence class will disagree on a subset of J_m that is $\notin u_m$, so that their difference is mapped by g into $g(k^{J_0})$). Hence the value of $g(a)$ modulo $g(k^{J_0})$ is determined by the images of a in the spaces k^{J_m}/u_m This gives the factorization (22). The one-one-ness of the factoring map follows from (41).

To get from this a result with a simpler statement, recall that a set I admits a nonprincipal $\mathrm{card}(k)^+$- complete ultrafilter only if its cardinality is greater than or equal to a measurable cardinal > card(k) [7,Proposition 4.2.7]. (We follow [7] in counting ℵ0 as a measurable cardinal. Thus, we write"uncountable measurable cardinal" for what many authors, e.g., [8, p.177], simply call a "measurable cardinal".)

Now uncountable measurable cardinals, if they exist at all, must be enormously large (cf. [8, Chapter 6,Corollary 1.8]). Hence for k infinite, it is a weak restriction to assume that I is smaller than all $\mathrm{card}(k)^+$-complete cardinals. Under that assumption, the $\mathrm{card}(k)^+$-complete ultrafilters u_m of Lemma 7 must be principal, determined by elements $i_m \in I$; so each nugget J_m contains a minimal nugget, the singleton $\{i_m\}$, and we may use these minimal nuggets in our decomposition (20). The statement of Lemma 7 then simplifies to the next result. (No such simplification is possible if k is finite, since then every ultrafilter is $\mathrm{card}(k)^+$-complete, and the only restriction we could put on card(I) that would force all $\mathrm{card}(k)^+$-complete ultrafilters to be principal would be finiteness; an uninteresting situation. So we now exclude the case offinitek.)

Corollary 8. Let k be an infinite field, I a set of cardinality less than every measurable cardinal > card(k)(if any exist), V a finite-dimensional k-vector space, and $g : k^I \to V$ a k-linear map. Then there existelements $i_1,\ldots,i_n \in I$ such that, writing $J_0 = I - \{i_1,\ldots,i_n\}$, we have

(45) Every element of $g(k^{J_0})$ is the image under g of an element having support precisely J_0.

In particular, applying this to $0 \in g(k^{J_0})$,

(46) There exists some $u = (u_i) \in \ker(g)$ such that $u_i = 0$ for only finitely many i (namely $i_1,\ldots,i_n$).

Since we have excluded the case where k is finite, the above corollary did not need condition (19), thatcard(k)≥ $\dim_k(V) + 2$. We end this section with a quick example showing that Lemma 7 does need that condition.Let k be any finite field, and I a subset of $k \times k$ consisting of one nonzero element from each of thecard(k) + 1 one-dimensional subspaces of that two-dimensional space (i.e., I is a set of representatives of the points of the projective line over k). Let $S \subseteq k^I$ be the two-dimensional subspace consisting of the restrictions to I of all k-linear functionals on $k \times k$. Since k^I is (card(k)+1)-dimensional, S can be expressed as the kernel of a linear map g from k^I to a (card(k)−1)-dimensional vector space V. By choice of I, every element of S = ker(g) has a zero somewhere on I, so $0 \in g(k^I)$ is not the imageunder g of an element having all of I for support. Hence (21) cannot hold with $J_0 = I$. If Lemma 7 were applicable, this would force the existence of a nonzero number of nuggets J_m. Since I is finite, the associated ultrafilters would be principal, corresponding to elements i_m such that all members of $S = \ker(g)$ were zero at i_m (by the one-one-ness of the last map of (22)). But this does not happen either: for every $i \in I$, there are clearly elements of S nonzero at i.Hence the conclusion of Lemma 7 does not hold for this g. Note that since $\dim_k(V) =$ card(k)−1, thecondition card(k) ≥ $\dim_k(V)$ + 2 fails by just 1.

On the other hand, fixing k and an infinite set I, and looking at how large V can be allowed to be, we see that for $V = k^{\aleph 0}$, projection of k^I to a countable subproduct gives a map $k^I \to V$ whose kernel has no elements of finite support; so we cannot allow $\dim_k(V)$ to reach $\dim_k(k^{\aleph 0})$. By the Erd″os-Kaplansky Theorem [11, Theorem IX.2, p.246], this equals card $(k)^{\aleph 0}$. Now if card(k) has the form $\lambda^{\aleph 0}$ for some λ, then card$(k)^{\aleph 0} =$ card(k); so in that case, Lemma 5 gives the weakest possible hypothesis on $\dim_k(V)$. Likewise, the hypothesis on $\dim_k(V)$ in Lemma 6 is optimal for countable k. But we don't know whether for general uncountable k, we can weaken the hypothesis $\dim_k(V)<$ card(k) of Lemma 5 all or part of the way to $\dim_k(V)<$ card $(k)^{\aleph 0}$.

Turning to our results on algebras over fields, let us mention that , Theorem combines the very weak hypothesis on card(I) in Theorem 9(iii) of this note with the hypothesis $\dim_k(B) \leq \aleph 0$, weaker than that of Theorem 9(iii), but imposes the additional condition that as an algebra, B satisfy "chain condition on almost direct factors" (defined there). That condition is automatic for finite-dimensional algebras, hence that result subsumes part (iii) of our present theorem. We do not know whether that chain condition can be dropped from the result of Incidentally, most of the results of do not exclude the case where card(I) is ≥ a measurable cardinal > card(k), but instead give, in that case, a conclusion in which factorization of $f: \Pi_I A_i \to B$ throug finitely many of the A_i is replaced by factorization through finitely many ultraproducts of the A_i with respect to $\mathrm{card}(k)^+$-complete ultrafilters. Though similar factorizations for a linear map $g: k^I \to B$ appear in Lemma 7 of this note, an apparent obstruction to carrying these over to results on algebra homomorphisms is that our proof of the latter applies the results of §§2-3 not just to a single linear map $g_a: k^I \to B$, but to one such map for each $a \in A = \Pi_I A_i$; and different maps yield different families of ultrafilters. However, one can get around this by choosing finitely many elements $a_1, \ldots, a_d \in A$ whose images under f span B, regarding them as together determining a map $g_{a_1,\ldots,a_d}: k^I \to B^d$, applying Lemma 7 to that map, and then showing that the image under f of any element in the kernels of all the resulting ultraproduct maps has zero product with the images of $a_1, \ldots, a_d \in A$, hence lies in Z(B). For the sake of brevity we have not set down formally a generalization of Theorem 9(iii) based on this argument. For other results on cardinality and factorization of maps on products, but of a somewhat different flavor

Acknowledgements:

We would like to thank our chairman Dr. Kancharla. Ramaiah garu and the many collegues who contributed in different ways to this work. This work was supported by The centre of Excellence in Mathematics and Physics, Prakasam Engineering college, Kandukur.

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