

“INTEGRALS INVOLVING APPELL’S FUNCTION OF MATRIX ARGUMENT IN COMPLEX CASE”

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Abstract: In this paper we investigated certain integral representation of Appell function F_1 , F_2 , and F_3 of real hermitian positive definite matrices. The results for scalar argument readily follow by taking the matrices of order unity. Some of the results obtained are believed to be new.

INTRODUCTION

Following the technique developed by Herz(1955) and Mathai(1991,1993) the authors define the Appell function F_1 , F_2 and F_3 of two Matrix arguments in terms of certain double integrals of matrix arguments. The definition of Appell function F_4 in terms of integrals is not given because a convenient integral is not available. These definition are useful in deriving several uninteresting, useful and new properties of Appell functions of matrix arguments. It is not out of place to mention that there are many practical problems in which Appell functions occur such as geometric probability time series model, queing models, engineering problems, etc. Exton(1976) has given a detailed account of statistical distribution where various special function of one and more variable are involved. Gamma, Beta and Dirichlet densities of matrix variates occur in the distributions of various test statistics in multivariable analysis and in the distribution connected with the concepts of generalized variance and canonical correlation matrices. These results can be seen in the monograph by **Mathai and Saxena(1978)**. Mathai and Saxena(1987) also discussed

various problems of statistical distribution where Lauricella function of several complex variables occur naturally, other useful applications of special function can be seen from the work of Saxena and Sethi (1973). Recently Mathai (1993) defined and established several integral representations for Lauricella function of real symmetric positive definite matrices.

In multivariate statistical theory series expansions of Herz hypergeometric function were developed by James, Constantine, Mathai, Sethi and others statistical distributions. We refer to James[1964], Muirhead[1983], and Takemura[1985], Mathai[1993], Sethi and Vyas[2001] for extensive surveys of the statistical literature. Although series expansions for hypergeometric function appear in the literature for both the real and complex fields due to James[1964].

The object of this paper is to derive certain integral representations for Appell function F_1 ; F_2 and F_3 of two matrix arguments.

APPELL FUNCTION F_1 OF MATRTRX ARGUMENT

The Appell function of matrix argument will be denoted by $F_1(\alpha, \beta, \beta', \gamma, \tilde{X}, \tilde{Y})$ analogous to the corresponding scalar case and is defined by the integral.

$$F_1(\alpha, \beta, \beta', \gamma, \tilde{X}, \tilde{Y}) = \frac{\tilde{\Gamma}_m(\gamma)}{\tilde{\Gamma}_m(\alpha)\tilde{\Gamma}_m(\gamma-\alpha)} \int_0^1 |\det \tilde{U}|^{\alpha-m} |\det(I-\tilde{U})|^{\gamma-\alpha-m} |\det(I-\tilde{U}\tilde{X})|^{-\beta} |\det(I-\tilde{U}\tilde{Y})|^{-\beta} d\tilde{U} \dots (1)$$

For

$$\operatorname{Re}(\gamma) > m - 1, \quad \operatorname{Re}(\alpha) > m - 1, \quad \operatorname{Re}(\gamma - \alpha) > m - 1$$

$$\tilde{X} = \tilde{X} > 0, \quad \tilde{Y} = \tilde{Y} > 0 \quad \|\det \tilde{X}\| < 1 \quad \|\det \tilde{Y}\| < 1$$

Certain integral representation of Appell function F_1 of matrix arguments are investigated in the form of following two theorem.

Theorem 1 :- For $\text{Re}(\delta) > m - 1$; $\text{Re}(\beta) > m - 1$; $\text{Re}(\beta') > m - 1$

$$\|\det \tilde{X}\| < 1, \|\det \tilde{Y}\| < 1$$

We have

$$F_1(\alpha, \beta; \beta'; \gamma'; \tilde{X}, \tilde{Y}) = \frac{\tilde{\Gamma}_m(\delta)\tilde{\Gamma}_m(\delta')}{\tilde{\Gamma}_m(\beta)\tilde{\Gamma}_m(\beta')\tilde{\Gamma}_m(\delta-\beta)\tilde{\Gamma}_m(\delta'-\beta')}$$

$$\int_0^1 \int_0^1 |\det \tilde{U}|^{\beta-m} |\det \tilde{V}|^{\beta'-m} |\det(\mathbf{I} - \tilde{U})|^{\delta-\beta-m}$$

$$|\det(\mathbf{I} - \tilde{V})|^{\delta-\beta'-m} F_1(\alpha, \delta, \delta'; \gamma'; \tilde{X}^{1/2} \tilde{U} \tilde{X}^{1/2}, \tilde{Y}^{1/2} \tilde{V} \tilde{Y}^{1/2}) d\tilde{U} d\tilde{V} \dots\dots\dots (2)$$

Proof :-

If we employ (1) on the R.H.S of (2) invert the order of integration, which is permissible due to the absolute convergence of the integrals involved in the process and integrate with the help of the integral.

$$2F_1(\alpha, \beta, \gamma'; \tilde{Z}^{1/2} \tilde{X} \tilde{Z}^{1/2}) = \frac{\tilde{\Gamma}_m(\gamma)}{\tilde{\Gamma}_m(\alpha)\tilde{\Gamma}_m(\gamma-\alpha)}$$

$$\int_0^1 |\det \tilde{U}|^{\alpha-m} |\det(\mathbf{I} - \tilde{U})|^{\gamma-\alpha-m} |\det(\mathbf{I} - \tilde{U}^{1/2} \tilde{Z}^{1/2} \tilde{X} \tilde{Z}^{1/2} \tilde{U}^{1/2})|^{-\beta} d\tilde{U} \dots\dots\dots (3)$$

Where

$$\text{Re}(\alpha) > (m - 1); \text{Re}(\gamma - \alpha) > (m - 1)$$

$$\text{Re}(\gamma) > (m - 1) \text{ and } \|\det \tilde{X}\tilde{Z}\| < 1$$

We also have

$$2F_1(\alpha, \beta; \beta; \tilde{Z}) = {}_1F_0(a; \tilde{Z}) = |\det(\mathbf{I} - \tilde{Z})|^{-\alpha} \Lambda \dots\dots (4)$$

$$\text{For } \|\det \tilde{Z}\| < 1$$

The Results (2) follow subject to the following condition.

$$\left\| \det \tilde{X}^{1/2} \tilde{U} \tilde{X}^{1/2} \right\| \leq \left\| \det \tilde{X} \right\| \cdot \left\| \det \tilde{U} \right\| \quad \text{Since } \left\| \det \tilde{V} \right\| < 1$$

and

$$\left\| \det \tilde{Y}^{1/2} \tilde{V} \tilde{Y}^{1/2} \right\| \leq \left\| \det \tilde{V} \right\| \cdot \left\| \det \tilde{Y} \right\| \leq \left\| \det \tilde{Y} \right\| \quad \text{Since } \left\| \det \tilde{V} \right\| < 1$$

The next theorem can be proved in the same way

Theorem 2 : If $\text{Re}(\beta) > (m - 1)$; $\text{Re}(\beta') > (m - 1)$

$$\left\| \det \tilde{X} \right\| \leq \left\| \det \tilde{Y} \right\| < 1$$

Then the following results hold

$$\begin{aligned} \tilde{\Gamma}_m(\beta) \tilde{\Gamma}_m(\beta') F_1(\alpha, \beta; \beta', \gamma; \tilde{X}, \tilde{Y}) &= \int_{\tilde{R} > 0} \int_{\tilde{S} > 0} \exp(-\text{tr}(\tilde{R} + \tilde{S})) \cdot \left| \det \tilde{R} \right|^{\beta-m} \left| \det(\tilde{S}) \right|^{\beta-m} \\ &\quad {}_1F_1(\alpha; \gamma; \tilde{X}^{1/2} \tilde{R} \tilde{X}^{1/2} + \tilde{Y}^{1/2} \tilde{S} \tilde{Y}^{1/2}) d\tilde{R} d\tilde{S} \end{aligned} \quad (5)$$

APPELL FUNCTION F_2 OF MATRIX ARGUMENT

The Appell function F_2 of matrix argument will be denoted by $F_2(\alpha, \beta, \beta'; \gamma, \gamma', \tilde{X}, \tilde{Y})$ analogous to the corresponding scalar case and is defined by the integral

$$\begin{aligned} F_2(\alpha, \beta, \beta'; \gamma, \gamma', \tilde{X}, \tilde{Y}) &= \frac{\tilde{\Gamma}_m(\gamma) \tilde{\Gamma}_m(\gamma')}{\tilde{\Gamma}_m(\beta) \tilde{\Gamma}_m(\beta') \tilde{\Gamma}_m(\gamma - \beta) \tilde{\Gamma}_m(\gamma' - \beta')} \\ &\quad \int_0^1 \int_0^1 \left| \det \tilde{U} \right|^{\beta-m} \left| \det \tilde{V} \right|^{\beta'-m} \left| \det(\mathbf{I} - \tilde{U}) \right|^{\gamma-\beta-m} \left| \det(\mathbf{I} - \tilde{V}) \right|^{\gamma'-\beta'-m} \end{aligned}$$

$$\left| \det(\mathbf{I} - \tilde{\mathbf{X}}^{1/2} \tilde{\mathbf{U}} \tilde{\mathbf{X}}^{1/2} - \tilde{\mathbf{Y}}^{1/2} \tilde{\mathbf{V}} \tilde{\mathbf{Y}}^{1/2}) \right|^{-\alpha} d\tilde{\mathbf{U}} d\tilde{\mathbf{V}} \tag{6}$$

Where $\operatorname{Re}(\beta, \beta'; \gamma - \beta, \gamma' - \beta') > m - 1$ $\tilde{\mathbf{X}} = \tilde{\mathbf{X}} > 0$ $\tilde{\mathbf{Y}} = \tilde{\mathbf{Y}} > 0$

$$\tilde{\mathbf{U}} = \tilde{\mathbf{U}}', \quad \tilde{\mathbf{V}} = \tilde{\mathbf{V}}' \quad \|\tilde{\mathbf{X}}\| + \|\tilde{\mathbf{Y}}\| < 1$$

It may be pointed out to the reader that in definition (6) one may use

$$\det(\mathbf{I} - \mathbf{L}' \tilde{\mathbf{U}} \mathbf{L} - \mathbf{M}' \tilde{\mathbf{V}} \mathbf{M})^{-\alpha} \tag{7}$$

where L, M are any matrices such that

$$\mathbf{L}\mathbf{L}' = \tilde{\mathbf{X}}, \quad \mathbf{M}\mathbf{M}' = \tilde{\mathbf{Y}}$$

This does not mean that the determinant in (6) is equal to the one in (7).

But if we write $\mathbf{H} = \tilde{\mathbf{X}}^{1/2} \mathbf{L}^{-1}$; $\mathbf{K} = \tilde{\mathbf{Y}}^{1/2} \mathbf{M}^{-1}$ then $\mathbf{H}'\mathbf{H} = \mathbf{I}$, $\mathbf{K}'\mathbf{K} = \mathbf{I}$

i.e H and K are orthogonal.

Substituting $\tilde{\mathbf{X}}^{1/2} \tilde{\mathbf{U}} \tilde{\mathbf{X}}^{1/2} = \mathbf{L}' \mathbf{H}' \tilde{\mathbf{U}} \mathbf{H} \mathbf{L}$,

$$\tilde{\mathbf{Y}}^{1/2} \tilde{\mathbf{V}} \tilde{\mathbf{Y}}^{1/2} = \mathbf{M}' \mathbf{K}' \tilde{\mathbf{V}} \mathbf{K} \mathbf{M} \text{ in}$$

(6) and transforming to $\tilde{\mathbf{R}} = \mathbf{H}' \tilde{\mathbf{U}} \mathbf{H} \mathbf{S} = \mathbf{K}' \tilde{\mathbf{V}} \mathbf{K}$ we obtain the definition in terms of (7)

Theorem 3 :- if

$$\operatorname{Re}(\alpha) > (m - 1), \quad \tilde{\mathbf{X}} = \tilde{\mathbf{X}} > 0 \quad \tilde{\mathbf{Y}} = \tilde{\mathbf{Y}} > 0 \quad \|\tilde{\mathbf{X}}\| + \|\tilde{\mathbf{Y}}\| < 1$$

the following results holds :-

$$F_1(\alpha, \beta, \beta'; \gamma; \gamma'; \tilde{\mathbf{X}}, \tilde{\mathbf{Y}}) = \frac{1}{\tilde{\Gamma}_m(\alpha)} \int_{\tilde{\mathbf{R}} > 0} |\det \tilde{\mathbf{R}}|^{\alpha-m} \exp(-\operatorname{tr} \tilde{\mathbf{R}}) {}_1F_1(\beta; \gamma; \tilde{\mathbf{X}}^{1/2} \tilde{\mathbf{R}} \tilde{\mathbf{X}}^{1/2}) {}_1F_1(\beta'; \gamma'; \tilde{\mathbf{Y}}^{1/2} \tilde{\mathbf{R}} \tilde{\mathbf{Y}}^{1/2}) d\tilde{\mathbf{R}} \tag{8}$$

Proof:- If we substitute for

$\left| \det(I - \tilde{X}^{1/2} \tilde{U} \tilde{X}^{1/2} - \tilde{Y}^{1/2} \tilde{V} \tilde{Y}^{1/2}) \right|^{-2}$ on the R.H.S of (6) interchange the order of integration which is permissible under the condition stated with the theorem and then integrate out \tilde{U} and \tilde{V} with the help of the result.

$${}_1F_1(\alpha, \beta; \tilde{X}^{1/2} \tilde{Z} \tilde{X}^{1/2}) = \frac{\tilde{\Gamma}_m(\beta)}{\tilde{\Gamma}_m(\alpha) \tilde{\Gamma}_m(\beta - \alpha)} \int_0^1 |\det \tilde{U}|^{\alpha - m} |\det(I - \tilde{U})|^{\beta - \alpha - m} \cdot \exp(\text{tr} \tilde{X}^{1/2} \tilde{Z} \tilde{X}^{1/2} \tilde{U}) d\tilde{U} \tag{9}$$

Valid for $\text{Re}(\beta) > (m - 1); \text{Re}(\beta) > (m - 1)$

$\text{Re}(\beta - \alpha) > (m - 1)$ and $\|\tilde{X}\tilde{Z}\| > 1$

Since $\text{tr} \tilde{X}^{1/2} \tilde{Z} \tilde{X}^{1/2} \tilde{U} = \text{tr} \tilde{U}^{1/2} \tilde{X}^{1/2} \tilde{Z} \tilde{X}^{1/2} \tilde{U}^{1/2}$ we obtain (8)

Theorem 4 :- If

$\text{Re}(\alpha), \text{Re}(\delta - \alpha) > m - 1$ and $\|\tilde{X}\| + \|\tilde{Y}\| < 1$

the following results holds :-

$$F_2(\alpha, \beta, \beta'; \gamma; \gamma', \tilde{X}, \tilde{Y}) = \frac{\tilde{\Gamma}_m(\delta)}{\tilde{\Gamma}_m(\alpha) \tilde{\Gamma}_m(\delta - \alpha)} \int_0^1 |\det \tilde{Z}|^{\alpha - m} |\det I - \tilde{Z}|^{\delta - \alpha - m} F_2(\delta, \beta, \beta'; \gamma, \gamma', \tilde{Z}^{1/2} \tilde{X} \tilde{Z}^{1/2}, \tilde{Z}^{1/2} \tilde{Y} \tilde{Z}^{1/2}) d\tilde{Z} \tag{10}$$

Proof :- writing F_2 as a double integral and using (6), we have

$$F_1(\delta, \beta, \beta'; \gamma, \gamma', \tilde{Z}^{1/2} \tilde{X} \tilde{Z}^{1/2}, \tilde{Z}^{1/2} \tilde{Y} \tilde{Z}^{1/2}) = \frac{\tilde{\Gamma}_m(\gamma) \tilde{\Gamma}_m(\gamma')}{\tilde{\Gamma}_m(\beta) \tilde{\Gamma}_m(\beta') \tilde{\Gamma}_m(\gamma - \beta) \tilde{\Gamma}_m(\gamma' - \beta')}$$

$$\int_0^1 \int_0^1 |\det \tilde{U}|^{\beta-m} |\det(\tilde{V})|^{\beta'-m} |\det(I - \tilde{U})|^{\gamma-\beta-m} |\det(I - \tilde{V})|^{\gamma'-m}$$

$$\left| \det(I - \tilde{Z}^{1/2} \tilde{X}^{1/2} \tilde{U} \tilde{X}^{1/2} \tilde{Z}^{1/2} - \tilde{Z}^{1/2} \tilde{Y}^{1/2} \tilde{V} \tilde{Y}^{1/2} \tilde{Z}^{1/2}) \right|$$

$$\left| \det(I - \tilde{Z}^{1/2} \tilde{X}^{1/2} \tilde{U} \tilde{X}^{1/2} \tilde{Z}^{1/2} - \tilde{Z}^{1/2} \tilde{Y}^{1/2} \tilde{V} \tilde{Y}^{1/2} \tilde{Z}^{1/2}) \right|^{-\delta} d\tilde{U} d\tilde{V} \quad (11)$$

substituting (11) on the R.H.S of (10) and integrating our \tilde{Z} with the help of the result (1) the result follows.

Theorem 5 :- If

$$\operatorname{Re}(\delta - \beta), \operatorname{Re}(\delta' - \beta') > m - 1$$

$$\operatorname{Re}(\gamma - \delta), \operatorname{Re}(\gamma' - \delta') > m - 1$$

$$\operatorname{Re}(\delta), \operatorname{Re}(\delta') > m - 1$$

$$\operatorname{Re}(\beta), \operatorname{Re}(\beta') > m - 1$$

$$\operatorname{Re}(\gamma'), \operatorname{Re}(\gamma') > m - 1$$

$$\|\tilde{X}\| < 1 \quad \text{and} \quad \|\tilde{Y}\| < 1$$

The following results hold.

$$F_2(\alpha, \beta, \beta'; \gamma, \gamma', \tilde{X} \tilde{Y}) = \frac{\tilde{\Gamma}_m(\delta) \tilde{\Gamma}_m(\delta')}{\tilde{\Gamma}_m(\beta) \tilde{\Gamma}_m(\beta') \tilde{\Gamma}_m(\delta - \beta) \tilde{\Gamma}_m(\delta' - \beta')}$$

$$\int_0^1 \int_0^1 |\det \tilde{U}|^{\delta-m} |\det \tilde{V}|^{\delta'-m} |\det(I - \tilde{U})|^{\delta-\beta-m} |\det(I - \tilde{V})|^{\delta'-\beta'-m}$$

$$F_2(\alpha, \delta, \delta'; \gamma, \gamma', \tilde{X}^{1/2} \tilde{U} \tilde{X}^{1/2}, \tilde{Y}^{1/2} \tilde{V} \tilde{Y}^{1/2}) d\tilde{U} d\tilde{V} \quad (12)$$

as pointed out by the reference, a proof of theorem (5) will follow directly from the result

$$\int_0^1 \tilde{U}^{\beta-m} |\det(\mathbf{I} - \tilde{U})|^{\delta-\beta-m} {}_1F_1(\delta; \gamma; \tilde{\mathbf{R}}^{1/2} \tilde{\mathbf{X}}^{1/2} \tilde{U} \tilde{\mathbf{X}}^{1/2} \tilde{\mathbf{R}}^{1/2}) d\tilde{U}$$

$$= \frac{\tilde{\Gamma}_m(\beta) \tilde{\Gamma}_m(\delta - \beta)}{\tilde{\Gamma}_m(\delta)} {}_1F_1(\beta; \delta; \tilde{\mathbf{R}}^{1/2} \tilde{\mathbf{X}} \tilde{\mathbf{R}}^{1/2}) \tag{13}$$

Theorem 6 :-

$$F_2(\alpha, \beta, \beta'; \gamma, \gamma', \tilde{\mathbf{X}}\tilde{\mathbf{Y}}) = \frac{\tilde{\Gamma}_m(\gamma) \tilde{\Gamma}_m(\gamma')}{\tilde{\Gamma}_m(\delta) \tilde{\Gamma}_m(\delta') \tilde{\Gamma}_m(\gamma - \delta) \tilde{\Gamma}_m(\gamma' - \delta')}$$

$$\int_0^1 \int_0^1 |\det \tilde{U}|^{\delta-m} |\det \tilde{V}|^{\delta'-m} |\det(\mathbf{I} - \tilde{U})|^{\gamma-\beta-m} |\det(\mathbf{I} - \tilde{U})|^{\gamma'-\delta'-m}$$

$$F_2(\alpha, \beta, \beta'; \delta, \delta', \tilde{\mathbf{X}}^{1/2} \tilde{U} \tilde{\mathbf{X}}^{1/2}, \tilde{\mathbf{Y}}^{1/2} \tilde{V} \tilde{\mathbf{Y}}^{1/2}) d\tilde{U} d\tilde{V} \tag{14}$$

Proof:- If we substitute for F_2 on the R.H.S of (14), it gives 4- fold integral as

$$\sum_1 = X_1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 |\det \tilde{U}|^{\beta-m} |\det \tilde{V}|^{\beta-m} |\det(\mathbf{I} - \tilde{U})|^{\delta-\beta-m}$$

$$|\det(\mathbf{I} - \tilde{V})|^{\delta'-\beta'-m} |\det \tilde{\mathbf{R}}|^{\delta-m} |\det \tilde{\mathbf{S}}|^{\delta'-m}$$

$$|\det(\mathbf{I} - \tilde{\mathbf{R}})|^{\gamma-\delta-m} |\det(\mathbf{I} - \tilde{\mathbf{S}})|^{\gamma'-\delta'-m}$$

$$\left| \det(\mathbf{I} - \tilde{\mathbf{X}}^{1/2} \tilde{\mathbf{R}}^{1/2} \tilde{U} \tilde{\mathbf{R}}^{1/2} \tilde{\mathbf{X}}^{1/2} - \tilde{\mathbf{Y}}^{1/2} \tilde{\mathbf{S}}^{1/2} \tilde{V} \tilde{\mathbf{S}}^{1/2} \tilde{\mathbf{Y}}^{1/2}) \right|^\alpha d\tilde{U} d\tilde{V} d\tilde{\mathbf{R}} d\tilde{\mathbf{S}} \tag{15}$$

Where

$$X_1 = \frac{\tilde{\Gamma}_m(\gamma) \tilde{\Gamma}_m(\gamma')}{\tilde{\Gamma}_m(\beta) \tilde{\Gamma}_m(\beta') \tilde{\Gamma}_m(\delta' - \beta') \tilde{\Gamma}_m(\delta - \beta) \tilde{\Gamma}_m(\gamma - \delta) \tilde{\Gamma}_m(\gamma' - \delta')}$$

Consider the transformation of \tilde{U} to $\tilde{\lambda}_1$ and \tilde{V} to $\tilde{\lambda}_2$ through

$$\tilde{\mathbf{R}}^{1/2} \tilde{U} \tilde{\mathbf{R}}^{1/2} = \tilde{\lambda}_1 \Rightarrow |\det \tilde{\mathbf{R}}|^m d\tilde{U} = d\tilde{\lambda}_1$$

$$\tilde{\mathbf{S}}^{1/2} \tilde{V} \tilde{\mathbf{S}}^{1/2} = \tilde{\lambda}_2 \Rightarrow |\det \tilde{\mathbf{S}}|^m d\tilde{V} = d\tilde{\lambda}_2$$

$$0 < \tilde{\lambda}_1 < \mathbf{R}, \quad 0 < \tilde{\lambda}_2 < \mathbf{S}, \text{ and}$$

$$\tilde{U} = \tilde{R}^{1/2} \tilde{\lambda}_1 \tilde{R}^{1/2}, \tilde{V} = \tilde{S}^{1/2} \tilde{\lambda}_2 \tilde{S}^{1/2}$$

Substituting the above transformation in (15) we observe that

$$\begin{aligned} \sum \delta_1 = \tilde{X}_1 \int_{\tilde{\lambda}_1} \int_{\tilde{\lambda}_2} \int_{\tilde{R}} \int_{\tilde{S}} & |\det \tilde{\lambda}_1|^{\beta-m} |\det \tilde{\lambda}_2|^{\beta'-m} \\ & \cdot |\det(\tilde{R} - \tilde{\lambda}_1)|^{\delta-\beta-m} |\det \tilde{S} - \tilde{\lambda}_2|^{\delta'-\beta'-m} \\ & \cdot |\det(\tilde{I} - \tilde{R})|^{\gamma-\delta-m} |\det(\tilde{I} - \tilde{S})|^{\gamma'-\beta'-m} \\ & \cdot |\det(\tilde{I} - \tilde{X} \tilde{\lambda}_1 - \tilde{Y} \tilde{\lambda}_2)|^\alpha d \tilde{\lambda}_1 - d \tilde{\lambda}_2 d \tilde{R} d \tilde{S} \end{aligned} \quad (16)$$

The integration is taken over $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{R}$ and \tilde{S} such that

$$0 < \tilde{\lambda}_1 < \tilde{R}, 0 < \tilde{\lambda}_2 < \tilde{S}, 0 < \tilde{R} < \tilde{I} \text{ and } 0 < \tilde{S} < 1 \text{ respectively}$$

Here $|\det(\tilde{R} - \tilde{\lambda}_1)| = |\det\{(\tilde{I} - \tilde{\lambda}_1) - (\tilde{I} - \tilde{R})\}|, |\det(\tilde{S} - \tilde{\lambda}_2)| = |\det\{(\tilde{I} - \tilde{\lambda}_2) - (\tilde{I} - \tilde{S})\}|$

Put $\tilde{I} - \tilde{R} = (\tilde{I} - \tilde{\lambda}_1)^{1/2} \tilde{Z}_1 (\tilde{I} - \tilde{\lambda}_1)^{1/2}$

$$d \tilde{R} = (\tilde{I} - \tilde{\lambda}_2)^{1/2} \tilde{Z}_1$$

$$\tilde{I} - \tilde{S} = (\tilde{I} - \tilde{\lambda}_2)^{1/2} \tilde{Z}_2 (\tilde{I} - \tilde{\lambda}_2)^{1/2}, d \tilde{S} = (\tilde{I} - \tilde{\lambda}_2)^m d \tilde{Z}_2,$$

The above substitution in (15) yield

$$\begin{aligned} \sum_1 = X_1 \int_0^1 \int_0^1 & |\det \tilde{\lambda}_1|^{\beta-m} |\det \tilde{\lambda}_2|^{\beta'-m} |\det(\tilde{I} - \tilde{\lambda}_1)|^{\gamma-\beta-m} \cdot |\det(\tilde{I} - \tilde{\lambda}_2)|^{\gamma'-\beta'-m} \\ & \left| \det(\tilde{I} - \tilde{X}^{1/2} \tilde{\lambda}_1 \tilde{X}^{1/2} - \tilde{Y}^{1/2} \tilde{\lambda}_2 \tilde{Y}^{1/2}) \right| d \tilde{\lambda}_1 d \tilde{\lambda}_2 \cdot \\ & \int_0^1 \int_0^1 |\det(\tilde{Z}_1)|^{\gamma-\delta-m} |\det \tilde{Z}_2|^{\gamma'-\delta'-m} |\det(\tilde{I} - \tilde{Z}_1)|^{\delta-\beta-m} \\ & |\det(\tilde{I} - \tilde{Z}_2)|^{\delta'-\beta'-m} d \tilde{Z}_1 d \tilde{Z}_2 \end{aligned} \quad (17)$$

Now interpreting (15) with the help of the result due Herz (1955)

$$\beta_m(\alpha, \beta) = \int_0^1 |\det(\tilde{X})|^{\delta-m} |\det(\tilde{I} - \tilde{X})|^{\beta-m} d \tilde{X}$$

$$= \frac{\tilde{\Gamma}_m(\delta)\tilde{\Gamma}_m(\beta)}{\tilde{\Gamma}_m(\alpha + \beta)} \tag{18}$$

Valid for $\text{Re}(\alpha) > (m - 1)$ and $\text{Re}(\beta) > (m - 1)$ the theorem is established.

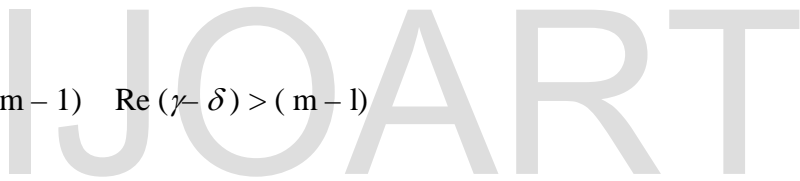
Alternative Proof :-

As pointed out by the referee, a simple proof of theorem (6) can be worked out as follows. Note that in theorem (3). Since ${}_1F_1$ (like ${}_1F_0$, ${}_2F_1$ etc.) is a function of the latent roots of the argument. We may also take $\tilde{\mathbf{R}}^{1/2}\tilde{\mathbf{X}}\tilde{\mathbf{R}}^{1/2}$, $\tilde{\mathbf{R}}^{1/2}\tilde{\mathbf{Y}}\tilde{\mathbf{R}}^{1/2}$ instead of $\tilde{\mathbf{X}}^{1/2}\tilde{\mathbf{R}}\tilde{\mathbf{X}}^{1/2}$ and $\tilde{\mathbf{Y}}^{1/2}\tilde{\mathbf{R}}\tilde{\mathbf{Y}}^{1/2}$. Hence the result (14) follows simply from the integral

$$\int_0^1 |\det(\tilde{\mathbf{U}})|^{\delta-m} |\det(\mathbf{I} - \tilde{\mathbf{U}})|^{\gamma-\delta-m} {}_1F_1(\beta; \delta; \tilde{\mathbf{R}}^{1/2}\tilde{\mathbf{X}}\tilde{\mathbf{R}}^{1/2}, \tilde{\mathbf{U}}\tilde{\mathbf{X}}\tilde{\mathbf{R}}^{1/2}, \tilde{\mathbf{R}}^{1/2}) d\tilde{\mathbf{U}}$$

$$= \frac{\tilde{\Gamma}_m(\delta)\tilde{\Gamma}_m(\gamma - \delta)}{\tilde{\Gamma}_m(\gamma)} {}_1F_1(\beta; \gamma; \tilde{\mathbf{R}}^{1/2}\tilde{\mathbf{X}}\tilde{\mathbf{R}}^{1/2}) \tag{19}$$

Where $\text{Re}(\delta) > (m - 1)$ $\text{Re}(\gamma - \delta) > (m - 1)$



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