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Formularization of Generalized Cantor Set and its Dynamical Behaviors

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ABSTRACT

In this article, we discuss the Cantor middle $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \cdots$ set, in general, Cantor middle $\frac{1}{2m-1}$ set, where $2 \le m < \infty$. We formularize the Generalized Cantor set and also show the dynamical behaviors of them using Mathematica programming.

Key Words: Formularization, Cantor set, and Cantor middle set.

1 INTRODUCTION

Cantor sets were discovered by the German Mathematician George Cantor in the late 19th to early 20th centuries (1845-1918). He introduced fractal which has come to be known as the Cantor set, or Cantor dust. The set has some interesting properties which have led to further research and discovery in fractals and chaos theory (Richard, 1996).
Fractal is defined by B. Mandelbrot is a shape made of parts similar to the whole in some way in the 1960's. Fractal is a geometric object that possesses the two properties: self-similar and non-integer dimensions. So a fractal is an object or quantity which displays self-similarity.

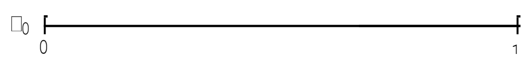
In ⊛ 2 we recall a Generalized Cantor set [2] (Cantor middle $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \cdots$ set, in general, Cantor middle $\frac{1}{2m-1}$ set, where $2 \le m < \infty$). In ⊛ 3 we define the functions of Cantor middle $\frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \cdots$ set, in general, Cantor middle $\frac{1}{2m-1}$ set and also show the dynamical behaviors of them use Mathematica programming.

2. GENERALIZATION OF CANTOR SET [2]

2.1 Definition

A non empty set $\Gamma \subset \mathbf{R}$ is called a Cantor set if
(a) Γ is closed and bounded.
(b) Γ contains no intervals.
(c) Every point in Γ is an accumulation point of Γ.

The Cantor middle $\frac{1}{3}$ set which is created by the following algorithm:
We start with the closed interval $[0,1]$. Call this set Γ_0.



Remove the middle open third. This leaves a new set, called Γ_1 which is $[0,\frac{1}{3}] \cup [\frac{2}{3},1]$.

Each iteration through the algorithm removes the open middle third from each segment of the previous iteration. Thus the next two sets would be

$$\Gamma_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1].$$

and

$$\Gamma_3 = [0,\frac{1}{27}] \cup [\frac{2}{27},\frac{1}{9}] \cup [\frac{2}{9},\frac{7}{27}] \cup [\frac{8}{27},\frac{1}{3}] \cup [\frac{2}{3},\frac{19}{27}] \cup [\frac{20}{27},\frac{7}{9}] \cup [\frac{8}{9},\frac{25}{27}] \cup [\frac{26}{27},1].$$

In general, after n times iterations, we obtain Γ_n which as follows

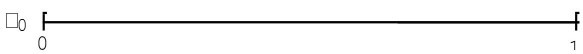
$$\Gamma_n = [0, \frac{1}{3^n}] \cup [\frac{2}{3^n}, \frac{3}{3^n}] \cup \cdots \cup [\frac{3^n-3}{3^n}, \frac{3^n-2}{3^n}]$$
$$\cup [\frac{3^n-1}{3^n}, 1], \text{where } n \geq 0.$$

The Cantor set is defined to be the set of the points that remain as the number of iterations tends to infinity.

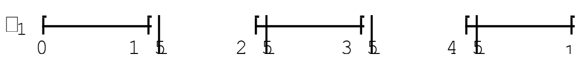
The Cantor middle $\frac{1}{3}$ set is the set $\Gamma = \bigcap_{n=0}^{\infty} \Gamma_n$.

The Cantor middle $\frac{1}{5}$ set which is created by the following algorithm:

We start with the closed interval $[0,1]$. Call this set Γ_0.

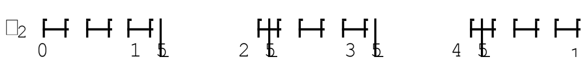


Remove the middle open interval $(1/5, 2/5)$ and $(3/5, 4/5)$. This leaves a new set, called Γ_1 which is $[0, \frac{1}{5}] \cup [\frac{2}{5}, \frac{3}{5}] \cup [\frac{4}{5}, 1]$.



Each iteration through the algorithm removes the open 2nd and 4th interval from each segment of the previous iteration. Thus the next set would be

$$\Gamma_2 = [0, \frac{1}{25}] \cup [\frac{2}{25}, \frac{3}{25}] \cup [\frac{4}{25}, \frac{1}{5}] \cup [\frac{2}{5}, \frac{11}{25}] \cup$$
$$[\frac{12}{25}, \frac{13}{25}] \cup [\frac{14}{25}, \frac{3}{5}] \cup [\frac{4}{5}, \frac{21}{25}] \cup [\frac{22}{25}, \frac{23}{25}] \cup [\frac{24}{25}, 1].$$



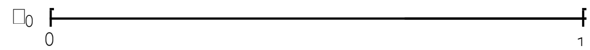
In general, after n times iterations, we obtain Γ_n which as follows

$$\Gamma_n = [0, \frac{1}{5^n}] \cup [\frac{2}{5^n}, \frac{3}{5^n}] \cup \cdots \cup [\frac{5^n-3}{5^n}, \frac{5^n-2}{5^n}]$$
$$\cup [\frac{5^n-1}{5^n}, 1], \text{ where } n \geq 0.$$

The Cantor middle $\frac{1}{5}$ set is the set $\Gamma = \bigcap_{n=0}^{\infty} \Gamma_n$.

The Cantor middle $\frac{1}{7}$ set which set is created by the following algorithm:

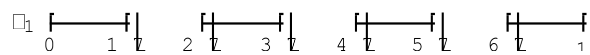
We start with the closed interval $[0,1]$. Call this set Γ_0.



Remove the middle open interval $(1/7, 2/7)$, $(3/7, 4/7)$, and $(5/7, 6/7)$.

This leaves a new set called Γ_1 which is

$[0, \frac{1}{7}] \cup [\frac{2}{7}, \frac{3}{7}] \cup [\frac{4}{7}, \frac{5}{7}] \cup [\frac{6}{7}, 1]$.



Each iteration through the algorithm removes the open 2nd, 4th, and 6th interval from each segment of the previous iteration. In general, after n times iterations, we obtain Γ_n which as follows

$$\Gamma_n = [0, \frac{1}{7^n}] \cup [\frac{2}{7^n}, \frac{3}{7^n}] \cup \cdots \cup [\frac{5^n-3}{5^n}, \frac{5^n-2}{5^n}]$$
$$\cup [\frac{5^n-1}{5^n}, 1], \text{ where } n \geq 0.$$

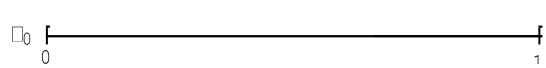
The Cantor middle $\frac{1}{7}$ set is the set $\Gamma = \bigcap_{n=0}^{\infty} \Gamma_n$.

From the above constructions, we can construct the Cantor middle $\frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \cdots$ set, in general,

the Cantor middle $\frac{1}{2m-1}$ set, where $2 \leq m < \infty$. If $0 < \frac{1}{2m-1} < 1$, where $2 \leq m < \infty$, then we can construct similar set called Cantor middle $\frac{1}{2m-1}$ set by removing an interval of each remaining interval whose length is $\frac{1}{2m-1}$ times the length of the remaining interval.

The Cantor middle $\frac{1}{2m-1}$ set which is created by the following algorithm:

We start with the closed interval $[0,1]$. Call this set Γ_0.



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Remove the middle open interval
$\left(\frac{1}{2m-1},\frac{2}{2m-1}\right)$, $\left(\frac{3}{2m-1},\frac{4}{2m-1}\right)$,
$\ldots,\left(\frac{2m-3}{2m-1},\frac{2m-2}{2m-1}\right)$, where $2\le m<\infty.$ In general, after n times iterations, we obtain Γ_n which as follows

$$[0,\frac{1}{(2m-1)^n}]\cup[\frac{2}{(2m-1)^n},\frac{3}{(2m-1)^n}]\cup\cdots$$
$$\cup[\frac{(2m-1)^n-3}{(2m-1)^n},\frac{(2m-1)^n-2}{(2m-1)^n}]\cup[\frac{(2m-1)^n-1}{(2m-1)^n},1],$$

where $n\ge 0$, and $m\ge 2.$

The Cantor middle $\frac{1}{2m-1}$ set is the set $\Gamma=\bigcap_{n=0}^{\infty}\Gamma_n.$

Hence the generalized Cantor set is Cantor middle $\frac{1}{2m-1}$ set, where $2\le m<\infty.$

2.2 Lemma [2]

If Γ_n is defined in Cantor middle $\frac{1}{2m-1}$ set, where $2\le m<\infty$, then there are m^n closed intervals in Γ_n and the length of each closed interval is $\left(\frac{1-\frac{1}{2m-1}}{2m-2}\right)^n$, where $2\le m<\infty.$ Also the combined length of the intervals in Γ_n is $\left(\frac{m}{2m-1}\right)^n$, where $2\le m<\infty$, which is approaches to zero as n approaches to infinity.

Proof: The proof of the Lemma is to be found in [2].

2.3 Proposition

The Cantor middle $\frac{1}{2m-1}$ set is a Cantor set, where $2\le m<\infty.$

Poof: The proof of the Proposition is to be found in [2].

3. FORMULARIZATION OF GENERALIZED CANTOR MIDDLE $\frac{1}{2m-1}$ SET AND ITS DYNAMICAL BEHAVIORS (MAIN RESULTS)

Let $\Gamma:\mathbf{R}\to\mathbf{R}$ be defined by

$$\Gamma(x)=\begin{cases}(2m-1)x & \text{if } x\le 1/m\\ 3-(2m-1)x & \text{if } 1/m<x\le 2/m\\ 5-(2m-1)x & \text{if } 2/m<x\le 3/m\\ 7-(2m-1)x & \text{if } 3/m<x\le 4/m\\ \vdots & \vdots\\ (2m-1)-(2m-1)x & \text{if } x>(m-1)/m\end{cases},$$

where $2\le m<\infty$, which is the function of the Cantor middle $\frac{1}{2m-1}$ set. The dynamical behaviors of this function for the different values of $m\ge 2$ are given below:

(1) If $m=2$, then we get the Cantor middle $\frac{1}{3}$ set which is defined by

$$\Gamma(x)=\begin{cases}3x & \text{if } x\le 1/2\\ 3-3x & \text{if } x>1/2\end{cases}$$

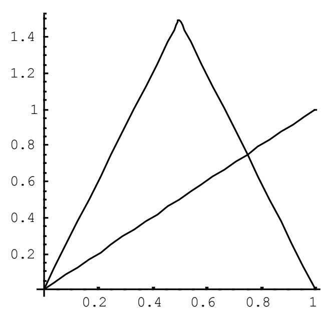

Fig. 3.1 Graph of Cantor middle $\frac{1}{3}$ set

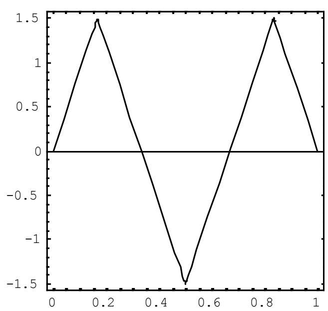

Fig. 3.2 Iteration=1

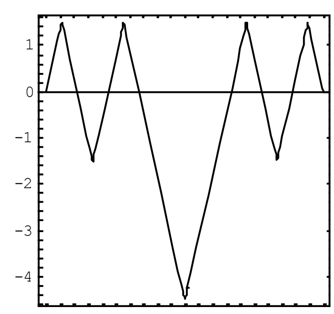

Fig. 3.3 Iteration=2

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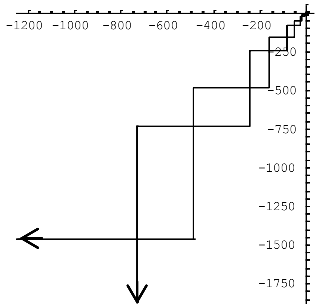


Fig. 3.4 Graph of $\Gamma^n (n=40),$ when $x>1, x<0.$

(i) The fixed points of Cantor middle $\frac{1}{3}$ set are 0 and $\frac{3}{4}.$

The ternary expansion of the fixed point $\frac{3}{4}$ is $0.202020\cdots.$

(ii) The above Fig. 3.4 shows that, if $x>1$ or $x<0,$ then $\Gamma^n \to -\infty$ as $n \to \infty.$

(iii) If x is any element of the removing open interval of Γ (i.e., $x \in (1/3, 2/3)\, or\, (7/9, 8/9)$), then $\Gamma^n \to -\infty$ as $n \to \infty.$ We get the following orbits for the point $x=1/2$ under Γ tends to negative infinity. That is, the orbits of $\Gamma^n \to -\infty$ as $n \to \infty$ are given below:

1.5 $\to$ -1.5 $\to$ -4.5 $\to$ -13.5 $\to$ -40.5 $\to$ -121.5 $\to$ -364.5 $\to \cdots$

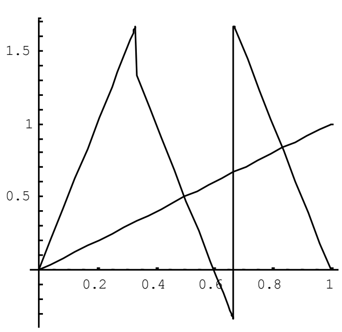
(iv) The points $x=\frac{3}{13}$ and $x=\frac{3}{28}$ lie on 3-cycles for Γ are given below:

0.692308 $\to$ 0.923077 $\to$ 0.230769 $\to$ 0.692308 $\to \ldots$

0.321429 $\to$ 0.964286 $\to$ 0.107143 $\to$ 0.321429 $\to \ldots$

(2) If $m=3,$ then we get the Cantor middle $\frac{1}{5}$ set which is defined by

$$\Gamma(x)=\begin{cases} 5x & \text{if } x \le 1/3 \\ 3-5x & \text{if } 1/3 < x \le 2/3 \\ 5-5x & \text{if } x > 2/3 \end{cases}$$



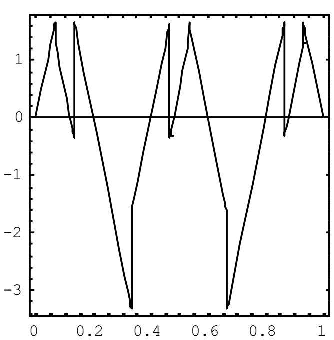


Fig. 3.5 Graph of Cantor middle $\frac{1}{5}$ set Fig. 3.6 Iteration=1

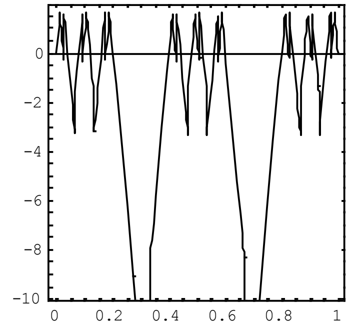


Fig. 3.7 Iteration=2

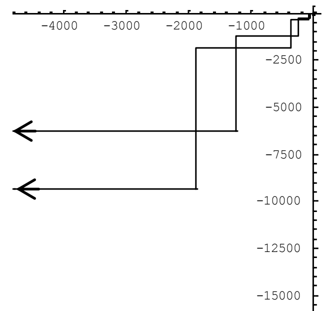


Fig. 3.8 Graph of $\Gamma^n (n=40),$ when $x>1, x<0.$

(i) The fixed points of Cantor middle $\frac{1}{5}$ set are $0,$ $\frac{1}{2},$ and $\frac{5}{6}.$

The ternary expansion of the fixed points $\frac{1}{2},$ and $\frac{5}{6}$ are $0.1111\cdots$ and $0.21111\cdots$ respectively.

(ii) The above Fig. 3.8 shows that, if $x>1$ or $x<0,$ then $\Gamma^n \to -\infty$ as $n \to \infty.$

(iii) If x is any element of the removing open interval of Γ (i.e., $x \in (1/5, 2/5)\, or\, (1/25, 2/25)$), then $\Gamma^n \to -\infty$ as $n \to \infty.$ We get the following orbits for the point $x=3/10$ under Γ tends to negative infinity. That is, the orbits of $\Gamma^n \to -\infty$ as $n \to \infty$ are given below:

1.5 $\to$ -2.5 $\to$ -12.5 $\to$ -62.5 $\to$ -312.5 $\to$ -1562.5 $\to \cdots$

(3) If $m=4,$ then we get the Cantor middle $\frac{1}{7}$ set which is defined by

$$\Gamma(x)=\begin{cases} 7x & \text{if } x \le 1/4 \\ 3-7x & \text{if } 1/4 < x \le 2/4 \\ 5-7x & \text{if } 2/4 < x \le 3/4 \\ 7-7x & \text{if } x > 3/4 \end{cases}$$

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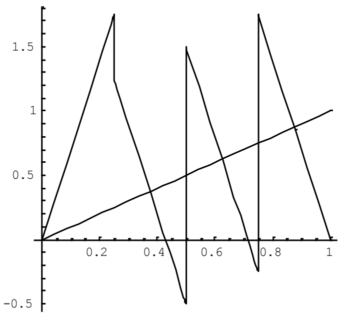

Fig. 3.9 Graph of Cantor middle $\frac{1}{7}$ set

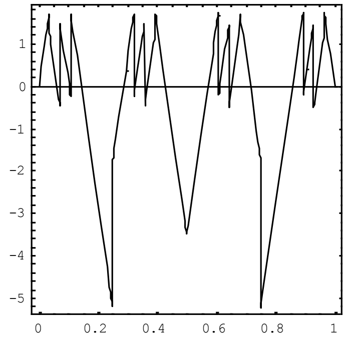

Fig. 3.10 Iteration=1

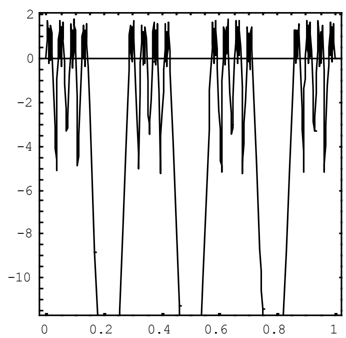

Fig. 3.11 Iteration=2

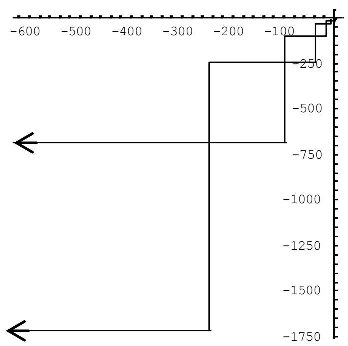

Fig. 3.12 Graph of $\Gamma^n (n=30)$, when $x>1, x<0$.

(i) The fixed points of Cantor middle $\frac{1}{7}$ set are $0, \frac{3}{8}, \frac{5}{8}$ and $\frac{7}{8}$. The ternary expansion of the fixed points $\frac{3}{8}, \frac{5}{8}$ and $\frac{7}{8}$ are $0.101010\cdots$, $0.121212\cdots$, and $0.212121\cdots$ respectively.

(ii) The above Fig. 3.12 shows that, if $x>1$ or $x<0$, then $\Gamma^n \to -\infty$ as $n \to \infty$.

(iii) If x is any element of the removing open interval of Γ (i.e., $x \in (1/7, 2/7)$ *or* $(1/49, 2/49)$), then $\Gamma^n \to -\infty$ as $n \to \infty$. We get the following orbits for the point $x = 3/14$ under Γ tends to negative infinity. That is, the orbits of $\Gamma^n \to -\infty$ as $n \to \infty$ are given below:

1.5 $\to$ -3.5 $\to$ -24.5 $\to$ -171.5 $\to$ -1200.5 $\to$ -8403.5 $\to \cdots$

(4) If $m=5$, then we get the Cantor middle $\frac{1}{9}$ set which is defined by

$$\Gamma(x) = \begin{cases} 9x & \text{if } x \le 1/5 \\ 3-9x & \text{if } 1/5 < x \le 2/5 \\ 5-9x & \text{if } 2/5 < x \le 3/5 \\ 7-9x & \text{if } 3/5 < x \le 4/5 \\ 9-9x & \text{if } x > 4/5 \end{cases}$$

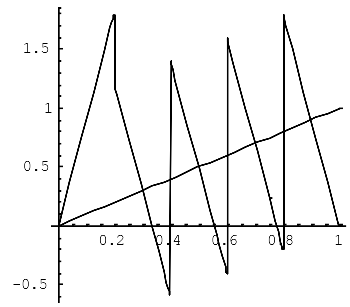

Fig. 3.13 Graph of Cantor middle $\frac{1}{9}$ set

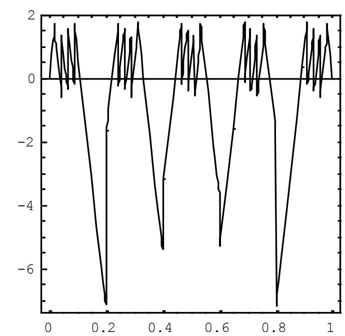

Fig. 3.14 Iteration=1

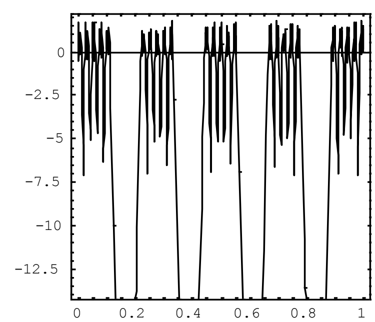

Fig. 3.15 Iteration=2

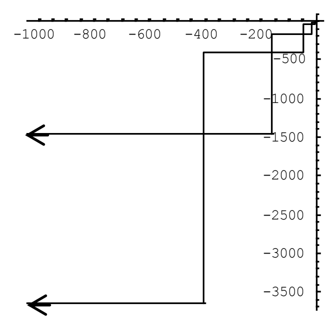

Fig. 3.16 Graph of $\Gamma^n (n=30)$, when $x>1, x<0$.

(i) The fixed points of Cantor middle $\frac{1}{9}$ set are $0, \frac{3}{10}, \frac{1}{2}, \frac{7}{10}$, and $\frac{9}{10}$. The ternary expansion of the fixed points $\frac{3}{10}$, $\frac{1}{2}$, $\frac{7}{10}$, and $\frac{9}{10}$ are $0.1111\cdots$, $0.2002200\cdots$, and $0.220022\cdots$ respectively.

(ii) The above Fig. 3.16 shows that, if $x > 1$ or $x < 0$, then $\Gamma^n \to -\infty$ as $n \to \infty$.

(iii) If x is any element of the removing open interval of Γ (i.e., $x \in (1/9, 2/9)$ *or* $(1/81, 2/81)$), then $\Gamma^n \to -\infty$ as $n \to \infty$. We get the following orbits for the point $x = 3/18$ under Γ tends to negative infinity. That is, the orbits of $\Gamma^n \to -\infty$ as $n \to \infty$ are given below:

$0.17 \to 1.5 \to -4.5 \to -40.5 \to -364.5 \to -3280.5 \to \cdots$

and so on we will get the above dynamical behavior of the function of the Cantor middle $\frac{1}{2m-1}$ set, where $2 \le m < \infty$.

CONCLUSION

We conclude that the dynamical behaviors of both Cantor middle $\frac{1}{3}$ set function and Generalized Cantor middle $\frac{1}{2m-1}$ set, where $2 \le m < \infty$ function are same such as if $x > 1$ or $x < 0$, then $\Gamma^n \to -\infty$ as $n \to \infty$ and also at the any point in the removing open interval of the Generalized Cantor middle $\frac{1}{2m-1}$ set, $\Gamma^n \to -\infty$ as $n \to \infty$.

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