

# Estimation of Parameters of Truncated Binormal Distribution By

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## Abstract :

In this paper we define doubly truncated binormal distribution. We have estimated the parameters of this distribution by method of moments. Mean deviation from mean in general form and recurrence relation has been obtained. Results regarding singly truncated binormal distribution have been shown as a particular case of doubly truncated binormal distribution.

## Keywords :

Doubly truncated binormal distribution, Method of moments, Recurrence relation.

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## 1 Introduction :

The binormal distribution was first introduced as the joint half Gaussian distribution by Gibbons and Mylorie (1973), in their study of experimental impurity profiles in ion-implanted amorphous targets. It has also been used by Toth and Szentimrey (1990) to model temperature distribution in climatological studies.

Johnson and Kotz (1970) have discussed about estimation of Truncated Normal distribution. S.B.Nabar and S.P. Barpande (2001) have given a note on the m.l.e. of the binormal distribution  $BIN(\sigma, \mu, \kappa\sigma)$ . Cohen, A.C. (1949) has discussed about the estimation of mean and standard deviation of truncated normal distribution.

We have defined doubly truncated binormal distribution and have obtained its mean and variance by method of moments in section – 2. In this section we have also obtained Mean deviation from mean. In section – 3 a general recurrence relation for the moments of distribution has been derived. Singly truncated binormal distribution and its estimation has been shown as a particular case of doubly truncated binormal distribution in section – 4.

## 2. Density function and estimation of parameters

The density function of binormal distribution  $BIN(\mu, \sigma_1, \sigma_2)$  is given as :

$$f(x) = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{1}{(\sigma_1 + \sigma_2)} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma_1}\right)^2\right] & , \quad x \leq \mu \\ \sqrt{\frac{2}{\pi}} \frac{1}{(\sigma_1 + \sigma_2)} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma_2}\right)^2\right] & , \quad x > \mu \end{cases}$$

Now, the density function of a doubly truncated binormal distribution is given as :

$$f(x) = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{1}{(\sigma_1 + \sigma_2)} \exp \left[ \frac{-1}{2} \left( \frac{x - \mu}{\sigma_1} \right)^2 \right] \left[ 2\sqrt{\frac{2}{\pi}} \frac{1}{(\sigma_1 + \sigma_2)} \int_A^\mu \exp \left( \frac{-1}{2} \left( \frac{x - \mu}{\sigma_1} \right)^2 \right) dx \right]^{-1} & A < x \leq \mu \\ \sqrt{\frac{2}{\pi}} \frac{1}{(\sigma_1 + \sigma_2)} \exp \left[ \frac{-1}{2} \left( \frac{x - \mu}{\sigma_2} \right)^2 \right] \left[ 2\sqrt{\frac{2}{\pi}} \frac{1}{(\sigma_1 + \sigma_2)} \int_\mu^B \exp \left( \frac{-1}{2} \left( \frac{x - \mu}{\sigma_2} \right)^2 \right) dx \right]^{-1} & \mu < x \leq B \end{cases} \dots\dots\dots(2.1)$$

(1) can be rewritten as

$$f(x) = \begin{cases} C_1^{-1} \sqrt{\frac{2}{\pi}} \frac{1}{(\sigma_1 + \sigma_2)} \exp \left[ \frac{-1}{2} \left( \frac{x - \mu}{\sigma_1} \right)^2 \right] & A < x \leq \mu \\ C_2^{-1} \sqrt{\frac{2}{\pi}} \frac{1}{(\sigma_1 + \sigma_2)} \exp \left[ \frac{-1}{2} \left( \frac{x - \mu}{\sigma_2} \right)^2 \right] & \mu < x \leq B \end{cases} \dots\dots\dots(2.2)$$

where

$$C_1 = 2 \sqrt{\frac{2}{\pi}} \frac{1}{(\sigma_1 + \sigma_2)} \int_A^\mu \exp \left[ \frac{-1}{2} \left( \frac{x - \mu}{\sigma_1} \right)^2 \right] dx$$

$$= \frac{4\sigma_1}{(\sigma_1 + \sigma_2)} \left[ \frac{1}{2} - \Phi(A^*) \right]$$

and

$$C_2 = 2 \sqrt{\frac{2}{\pi}} \frac{1}{(\sigma_1 + \sigma_2)} \int_\mu^B \exp \left[ \frac{-1}{2} \left( \frac{x - \mu}{\sigma_2} \right)^2 \right] dx$$

$$= \frac{4\sigma_2}{(\sigma_1 + \sigma_2)} \left[ \Phi(B^*) - \frac{1}{2} \right]$$

with

$$A^* = \frac{A - \mu}{\sigma_1}, \quad B^* = \frac{B - \mu}{\sigma_2} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \quad \dots\dots\dots (2.3)$$

### Estimation of parameters

Mean and variance of the distribution has been obtained using method of moments.

$$\begin{aligned} \bar{\mu} = E(x) &= \int_A^B f(x) dx = \int_A^{\mu} x f(x) dx + \int_{\mu}^B x f(x) dx \\ &= \mu + \frac{\sigma_1}{2} g_1(A^*) + \frac{\sigma_2}{2} g_1(B^*) \end{aligned} \dots\dots\dots (2.4)$$

and

$$\begin{aligned} E(x^2) &= \int_A^B x^2 f(x) dx \\ &= \mu^2 + \frac{\sigma_1^2 + \sigma_2^2}{2} + \sigma_1 \mu g_1(A^*) + \sigma_2 \mu g_1(B^*) + \frac{\sigma_1^2}{2} g_2(A^*) + \frac{\sigma_2^2}{2} g_2(B^*) \end{aligned} \dots\dots\dots (2.5)$$

Hence, Variance of  $x$  is given by

$$V(x) = \frac{\sigma_1^2 + \sigma_2^2}{2} + \frac{\sigma_1^2}{4} (2g_2(A^*) - g_1(A^*)) + \frac{\sigma_2^2}{4} (2g_2(B^*) - g_1(B^*)) - \frac{\sigma_1 + \sigma_2}{2} g_1(A^*)g_1(B^*) \dots\dots\dots (2.6)$$

where

$$g_1(x) = \frac{z(x) - 1/\sqrt{2\pi}}{1/2 - \Phi(x)}, \quad g_2(x) = \frac{x z(x)}{1/2 - \Phi(x)} \quad \text{and} \quad z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

### Mean deviation from mean :

In this section, we have obtained mean deviation from mean for the doubly truncated binormal distribution

Mean deviation from mean ( $\bar{\mu}$ ) is given as

$$\begin{aligned} M = E(x - \bar{\mu}) &= \int_A^B |x - \bar{\mu}| f(x) dx \\ &= \frac{\sigma_1}{2} \mu_1^* g_1(\mu_1^*, A^*) + \frac{\sigma_2}{2} \mu_2^* g_1(\mu_2^*, B^*) \\ &\quad + \frac{\sigma_1}{2} g_2(\mu_1^*, A^*) + \frac{\sigma_2}{2} g_2(\mu_2^*, B^*) \end{aligned} \dots\dots\dots (2.7)$$

where

$$g_1(x, y) = \frac{\Phi(x) - \Phi(y)}{1/2 - \Phi(y)}, \quad g_2(x, y) = \frac{z(x) - z(y)}{1/2 - \Phi(y)} \quad \text{and} \quad \mu_1^* = \frac{\bar{\mu} - \mu}{\sigma_1}, \quad \mu_2^* = \frac{\bar{\mu} - \mu}{\sigma_2}$$

### 3. Recurrence relation

The general recurrence relation for the moments of doubly truncated binormal distribution is given as

$$\begin{aligned} \mu'_r &= \int_A^B x^r f(x) dx = \int_A^\mu x^r f_1(x) dx + \int_\mu^B x^r f_2(x) dx \\ &= \mu'_{1r} + \mu'_{2r} \end{aligned}$$

$$\text{where } \mu'_{1r} = \frac{\mu}{\sqrt{2\pi}} \mu'_{1(r-1)} + \sigma_1 A^{r-1} z(A^*) - \frac{\sigma_1}{\sqrt{2\pi}} \mu^{r-1} - \frac{\sigma_1^2(r-1)}{\sqrt{2\pi}} \mu'_{1(r-2)}$$

$$\text{and } \mu'_{2r} = \frac{\mu}{\sqrt{2\pi}} \mu'_{2(r-1)} - \sigma_2 B^{r-1} z(B^*) + \frac{\sigma_2}{\sqrt{2\pi}} \mu^{r-1} - \frac{\sigma_2^2(r-1)}{\sqrt{2\pi}} \mu'_{2(r-2)}$$

$$\begin{aligned} \therefore \mu'_r &= \frac{\mu}{\sqrt{2\pi}} \mu'_{r-1} - \frac{\mu^{r-1}}{\sqrt{2\pi}} (\sigma_1 - \sigma_2) + \sigma_1 A^{r-1} z(A^*) - \sigma_2 B^{r-1} z(B^*) \\ &\quad - \frac{\sigma_1^2(r-1)}{\sqrt{2\pi}} \mu'_{1(r-2)} - \frac{\sigma_2^2(r-1)}{\sqrt{2\pi}} \mu'_{2(r-2)} \end{aligned} \quad \dots\dots\dots (3.1)$$

### 4. Singly truncated binormal distribution as a particular case.

In this section we obtain the results of singly truncated binormal distribution as a particular case of doubly truncated binormal distribution.

In (2.1) by taking  $B = \infty$ , lower truncated binormal distribution can be obtained as a particular case of doubly truncated binormal distribution. Its p.d.f is given by

$$f(x) = \begin{cases} C_1^{-1} \frac{1}{\sqrt{\pi}} \frac{1}{(\sigma_1 + \sigma_2)} \exp\left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma_1}\right)^2\right] & , A < x \leq \mu \\ \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma_2}\right)^2\right] & , \mu < x \leq \infty \end{cases} \quad \dots\dots\dots (4.1)$$

The Mean and Variance of the above distribution are given by

$$\begin{aligned} \text{Mean} &= \mu + \frac{\sigma_1}{2} g_1(A^*) + \frac{\sigma_2}{\sqrt{2\pi}} \\ \text{Variance} &= \frac{\sigma_1^2 + \sigma_2^2}{2} + \frac{\sigma_1^2}{4} (2g_2(A^*) - g_1(A^*)) - \frac{\sigma_2^2}{2\sqrt{2\pi}} - \frac{\sigma_1\sigma_2}{\sqrt{2\pi}} g_1(A^*) \end{aligned} \dots\dots\dots (4.2)$$

Similarly in (2.2) by taking  $A = -\infty$ , upper truncated binormal distribution can be obtained as a particular case of doubly truncated binormal distribution. Its p.d.f is given by

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma_1}\right)^2\right] & , -\infty < x < \mu \\ C_2^{-1} \sqrt{\frac{2}{\pi}} \frac{1}{\sigma_1 + \sigma_2} \exp\left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma_2}\right)^2\right] & , \mu < x < B \end{cases} \dots\dots\dots (4.3)$$

The mean and variance of the above distribution are given as

$$\begin{aligned} \text{Mean} &= \mu + \frac{\sigma_1}{\sqrt{2\pi}} + \frac{\sigma_2}{2} g_1(B^*) \\ \text{Variance} &= \frac{\sigma_1^2 + \sigma_2^2}{2} + \frac{\sigma_2^2}{4} (2g_2(B^*) - g_1(B^*)) - \frac{\sigma_1^2}{2\sqrt{2\pi}} - \frac{\sigma_1\sigma_2}{\sqrt{2\pi}} g_1(B^*) \end{aligned} \dots\dots\dots (4.4)$$

**Note :** We have also written a paper on “Estimation of Parameters of Truncated Bivariate Binormal distribution and have send it for publication.

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