ISSN 2278-7763

EIGEN VALUES FOR SIGNED BRAUER ALGEBRA

A.Vidhya and A.Tamilselvi

E-mail: vidhyamath@gmail.com

ABSTRACT

This paper determines the eigenvalues and eigenspaces of certain symmetric matrices $T_{m,k}(x)^{[\lambda]}$ for some values of m and k. For each m,k signed partial l-factor, the matrix $T_{m,k}(x)^{[\lambda]}$ with entries <.> which gives a complete structure of signed Brauer algebra $S_f^{(x)}$.

KeyWords

Eigenvalues, Eigenspaces, signed Braueralgebra, matix rings, representations, radicals.

1.Introduction

The signed Brauer algebra $S_f^{(x)}$ which is a centralizer of algebras of direct product of orthogonal groups over the field of real numbers $\mathbb{R}$ [5]. In this paper, we define the signed Brauer algebra in terms of signed 1-factor and signed 1-factor in terms of m,k signed partial l-factor. Also, we define certain symmetric matrices $T_{m,k}(x)^{[\lambda]}$ for those m,k signed partial l-factor $P_{m,k}$. We analyse the irreducible representations of $S_f^{(x)}$ by determining the quotient $H_f^{(x)}(2k)$ by its radical where the radical of $S_f^{(x)}$ must lie in the nullspace of $T_{m,k}(x)^{[\lambda]}$. We also find the eigenvalues and eigenspaces of certain symmetric matrices $T_{m,k}(x)^{[\lambda]}$ using the representation theory of the hyper-octahedral group of type B_n.

2.The structure of $S_f^{(x)}$

Definition 2.1. A signed 1-factor on $2f$ vertices is a signed diagram with $2f$ vertices and f signed edges such that each vertex is incident to exactly one edge where each edge is labeled by the edge with arrow or the edge without arrow. An edge without arrow is called positive edge and an edge with arrow is called negative edge.

Let P_f denote the set of all signed 1-factors on $2f$ vertices. Let V_f be the vector space with basis P_f. A signed 1-factor $\delta \in P_f$ will often be drawn as a diagram having two rows of $2f$ vertices each, the vertices $1,2,\ldots,f$ in a top row denoted by $t(\delta)$ and the vertices $f+1, f+2, \ldots, 2f$ in a bottom row denoted by $b(\delta)$. A signed edge of δ joining two vertices in $t(\delta)$ or two vertices in $b(\delta)$ is called a signed horizontal edge. A signed edge of δ joining a vertex in $t(\delta)$ to a vertex in $b(\delta)$ will be called a signed vertical edge. We let $V_f(2k)$ denote the subspace of V_f spanned by all $\delta \in P_f$ with $h(\delta) \geq 2k$ where $h(\delta)$ is the number of signed horizontal edges in P_f.

Let δ_1 and δ_2 be l-factors in P_f. The graph $U(\delta_1, \delta_2)$ with $3f$ vertices obtained by identifying the bottom row of δ_1 with the top row of δ_2.

It is easy to check, for any δ_1 and δ_2, and that $U(\delta_1,\delta_2)$ consists of exactly f signed paths $P_1, P_2, \ldots, P_f$ where signed path P_i is a path having signed edges and some number $\gamma(U(\delta_1,\delta_2))$ of signed cycles $C_1, \ldots, C_{\gamma(U(\delta_1,\delta_2))}$ where signed cycle C_i is a cycle having signed edges along the cycle satisfying:

(1) The endpoints of the signed paths P_i lie in the set $t(\delta_1) \cup b(\delta_2)$.
(2) Each signed cycle C_i is of even length and consists entirely of vertices in the set $b(\delta_1) = t(\delta_2)$.

Definition 2.2. Let δ_1 and δ_2 be l-factors in P_f. Define the braid of δ_1 over δ_2 denoted $\beta(\delta_1,\delta_2)$ to be the l-factor with top row $t(\delta_1)$ and bottom row $b(\delta_2)$ and with vertices u and v adjacent if and only if there is a signed path P_i in $U(\delta_1,\delta_2)$ joining u to v and an edge joining u to v is labeled by the product of the labels along the path P_i. Define algebras $S_f^{(x)} = (V_f, \circ)$ to be the $\mathbb{R}$-algebras with vector space bases V_f and multiplication of l-factors given by

$$\left(\text{in } S_f^{(x)}\right) \delta_1 \circ \delta_2 = x^{\gamma(U(\delta_1,\delta_2))}\beta(\delta_1,\delta_2)$$

where each cycle C_i in $U(\delta_1,\delta_2)$ is labeled by the product of the labels along the cycle C_i. This algebra $S_f^{(x)}$ is called the signed Brauer algebra [4].

The subspace $V_f(2k)$ spanned by all signed 1-factors of $2f$ vertices with atleast $2k$ signed horizontal edges spans an ideal $S_f^{(x)}(2k)$ of $S_f^{(x)}$. To describe the structure of the quotients $S_f^{(x)}(2k)/ S_f^{(x)}(2k+2)$ in terms of the eigenvalues and eigenspaces of certain matrices. Let $H_f^{(x)}(2k)$ denote the quotient $S_f^{(x)}(2k)/ S_f^{(x)}(2k+2)$.

Definition 2.3. A m,k signed partial l-factor is a graph with $m+2k$ vertices and k signed lines and m free vertices. Let $P_{m,k}$ denote the set of m,k signed partial l-factors, and let $V_{m,k}$ be the real vector space with basis $P_{m,k}$.

A labeled m,k signed partial l-factor is a graph with $m+2k$ vertices, has exactly k signed edges and m signed free vertices. Let us now define $\pi \in \tilde{S}_m$ by $\pi(i) = (\tau(i), \sigma(i))$ where $\sigma \in S_m$ and $\tau: \underline{m} \to \underline{2}$, $\underline{m}$ denotes the set $\{1,2,\ldots,m\}$ and $\underline{2}$ denotes the set $\{\pm 1\}$.

A signed path P_i joining the free vertex u to free vertex v is labeled by the product of the labels along the path P_i.

A signed cycle C_i is labeled by the product of the labels along the cycle C_i.

Let f_1 and f_2 be m,k signed partial l-factors with $\alpha_1 < \alpha_2 < \cdots < \alpha_m$ the free vertices of f_1 and $\beta_1 < \beta_2 < \cdots < \beta_m$ the free vertices of f_2. The union of f_1 and f_2 is a signed graph consisting of some number $\gamma(f_1,f_2)$ of disjoint signed cycles together with m disjoint signed paths $P_1, \ldots, P_m$ whose endpoints are in the set $\{\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_m\}$. Define an inner product $\langle f_1, f_2 \rangle$ on $V_{m,k}$ as follows. Given f_1, f_2 as above:

1. If any signed path P_i joins a α_j to a α_i (or equivalently a β_j to a β_i) then $\langle f_1, f_2 \rangle = 0$.
2. If P_i joins β_i to $\alpha_{\sigma i}$ and labeled by $\tau(i)$ then $\langle f_1, f_2 \rangle = x^{\gamma(f_1,f_2)}\pi$ where $\pi \in \tilde{S}_m$, $\tilde{S}_m$ is the hyperoctahedral group of type B_n. Note that $\langle f_1, f_2 \rangle = \langle f_2^*, f_1^* \rangle$, where $*$ is the anti-isomorphism defined on the algebra $\mathbb{R}\tilde{S}_m$ by $\sigma \to \sigma^{-1}$.

Proposition 2.4. Let $f = m+2k$. Then the quotient $H_f^{(x)}(2k)$ is isomorphic as algebra to $(V_{m,k} \otimes V_{m,k} \otimes \mathbb{R}\tilde{S}_m, \cdot)$, where $(a \otimes b \otimes z) \cdot (c \otimes d \otimes y) = a \otimes d \otimes (z\ \langle b,c \rangle\ y)$.

Proof. As a vector space $H_f^{(x)}(2k)$ has basis the set of all signed l-factor with exactly $2k$ signed horizontal edges.

Define the linear map ϕ: $V_{m,k} \otimes V_{m,k} \otimes \mathbb{R}\tilde{S}_m \to \mathrm{H}_f^{(x)}(2k)$, in the following way. Given $f_1, f_2 \in P_{m,k}$ with free vertices $\alpha_1 < \alpha_2 < \cdots <$

α_m and $\beta_1 < \beta_2 < \cdots < \beta_m$ and given $\sigma \in \tilde{S}_m$ define $\phi(f_1 \otimes f_2 \otimes \sigma)$ to be the signed l-factor of $2f$ vertices with

1. a signed horizontal edge joining i to j in the top row if and only if i and j are adjacent in f_1;
2. a signed horizontal edge joining $(f+i)$ to $(f+j)$ in the bottom row if and only if i and j are adjacent in f_2;
3. a signed vertical edge joining α_i to $\beta_{\sigma i}$ and labeled by $\tau(i)$ where $\pi = (\tau, \sigma) \in \tilde{S}_m$, $\sigma \in S_m$.

By the above construction of ϕ, ϕ is 1-1 and onto and hence it is a vector space isomorphism of $V_{m,k} \otimes V_{m,k} \otimes \mathbb{R}\tilde{S}_m$, onto $H_f^{(x)}(2k)$. It remains to show that ϕ is multiplicative. It is enough to prove $\phi(x \cdot y) = \phi(x) \circ \phi(y)$, $x, y \in V_{m,k} \otimes V_{m,k} \otimes \mathbb{R}\tilde{S}_m$.

Let $x = a \otimes b \otimes \pi_1$ and $y = c \otimes d \otimes \pi_2$, $a, b, c, d \in P_{m,k}$, $\pi_1, \pi_2 \in \tilde{S}_m$.

Let a, b, c, d be m,k signed partial l-factor with free vertices $\alpha_1 < \alpha_2 < \cdots < \alpha_m$, $\beta_1 < \beta_2 < \cdots < \beta_m$, $\gamma_1 < \gamma_2 < \cdots < \gamma_m$, $\delta_1 < \delta_2 < \cdots < \delta_m$ respectively.

By the $\circ$ defined above, $d_1 \circ d_2 = x^\gamma d_3$, $d_1, d_2 \in H_f^{(x)}(2k)$, $d_3 \in S_f^{(x)}(2k)$.

Case 1. Suppose there is a signed path joining α_i to $\alpha_{\sigma i}$ in $U(\delta_1, \delta_2)$ then $\phi(a \otimes b \otimes \pi_1) \circ \phi(c \otimes d \otimes \pi_2) = 0 = \phi((a \otimes b \otimes \pi_1) \cdot (c \otimes d \otimes \pi_2))$. Therefore $\phi(x \cdot y) = \phi(x) \circ \phi(y)$.

Case 2. Suppose there is a signed path joining α_i to $\delta_{\sigma i}$ in $U(\delta_1, \delta_2)$ then $\phi(x \cdot y) = x^\gamma d_3 = \phi(x) \circ \phi(y)$.

Hence ϕ is an algebra isomorphism.

Definition 2.5. A bi-partition $[\lambda]$ of n denoted by $[\lambda] \vdash n$, is 2 -tuple of partitions $[\lambda] = (\lambda^{(1)}, \lambda^{(2)})$ such that $\sum_{i=1}^{2} |\lambda^{(i)}| = n$.
Definition 2.6. Given a bi-partition $[\lambda]$ of n, we mean the Young diagram of $[\lambda]$ filled with the entries containing $\{(\pm 1, i)\}$ in such a way that the entries containing exactly one of $\{(\pm 1, i)\}$ is called bi-tableau t.
Definition 2.6. For each bi-partition $[\lambda]$, the Young subgroup $S_{[\lambda]} = S_{\lambda^{(1)}} \times S_{\lambda^{(2)}}$.
Definition 2.7. Given a bi-tableau t of shape $[\lambda] \vdash n$, the set of all elements in $\tilde{S}_n$, which leaves the rows of (2)nd-residue stable upto sign change and (1)st-residue stable, is a subgroup of $\tilde{S}_n$, called the row stabilizer and denoted by R_t,

$$R_t \simeq S_{\lambda_1^{(1)}} \times \cdots \times S_{\lambda_{q_1}^{(1)}} \times \tilde{S}_{\lambda_1^{(2)}} \times \cdots \times \tilde{S}_{\lambda_{q_r}^{(2)}}$$

where $\lambda_i^{(j)}$ is the length of the i^{th} row in j^{th} residue.
Definition 2.8. Given a bi-tableau t of shape $[\lambda] \vdash n$, the set of all elements in $\tilde{S}_n$, which leaves the columns of (2)nd-residue stable and (1)st-residue stable upto sign change, is a subgroup of $\tilde{S}_n$, called the column stabilizer and denoted by C_t,

$$C_t \simeq \tilde{S}_{\lambda_1'^{(1)}} \times \cdots \times \tilde{S}_{\lambda_{q_0}'^{(1)}} \times S_{\lambda_1'^{(2)}} \times \cdots \times S_{\lambda_{q_r}'^{(2)}}$$

where $\lambda_i'^{(j)}$ is the length of the i^{th} column in j^{th} residue.
Note $R_t = \prod_{i=1}^{r} R_t^{(i)}$ and $C_t = \prod_{i=1}^{r} C_t^{(i)}$.
Definition 2.9. For any tableau t, $\kappa_t = \prod_{i=1}^{r} \sum_{\pi \in C_t^{(i)}} \epsilon_i(\pi)\pi$ and $A_t = \kappa_t\{t\}$.

To describe the structure of the ring $D_f^{(x)}(2k)$ in terms of the eigenvalues of certain matrices. We begin by recalling several facts from the representation theory of the the hyper-octahedral group of type B_n. For each partition $[\lambda]$ of m, let $S^{[\lambda]}$ denote the Specht module corresponding to $[\lambda]$ and let $d_{[\lambda]}$ denote the dimension of $S^{[\lambda]}$.

Fact 1. There exists in group algebra of the hyper-octahedral $\hat{S}_m$ a unique minimal 2-sided ideal $\hat{S}_m$of dimension $(d_{[\lambda]})^2$ which can be written as direct sums $\hat{S}_m^{[\lambda]} = I_1 \oplus \cdots \oplus I_{d_{[\lambda]}}$, $\hat{S}_m^{[\lambda]} = J_1 \oplus \cdots \oplus J_{d_{[\lambda]}}$ where each I_i is a left ideal of $\hat{S}_m$ for which multiplication on the left gives a representation isomorphic to $S^{[\lambda]}$ and each J_i is a right ideal of $\hat{S}_m$ for which right multiplication is isomorphic to $S^{[\lambda]}$.
Fact 2. The ideal $\hat{S}_m^{[\lambda]}$ considered as a vector space of linear transformations of $S^{[\lambda]}$ is the full matrix algebra End($S^{[\lambda]}$).
Definition 2.10. Let $T_{m,k}(x)^{[\lambda]}$ be the $(pd_{[\lambda]})$-by-$(pd_{[\lambda]})$ matrix which is p-by-p blocks of $d_{[\lambda]}$-by-$d_{[\lambda]}$ matrices. The matrices in the each block are indexed by pairs of m, k signed partial l-factors with entries $< b, c >$, where $b, c \in P_{m,k}$ which is the Gram matrix of the $H_f^{(x)}(2k)$.

Let $N^{[\lambda]}$ and $R^{[\lambda]}$ denote the nullspace and range of $T_{m,k}(x)^{[\lambda]}$, respectively. Recall that if $< b, c >= x^\gamma \sigma$ then $< c, b >= x^\gamma \sigma^{-1}$. So the matrix $T_{m,k}(x)^{[\lambda]}$ is symmetric. Choose a basis $u^{(1)}, \ldots, u^{(n)}$ for $N^{[\lambda]}$ and an orthonormal basis of eigenvectors $v^{(1)}, \ldots, v^{(r)}$ for the nonzero eigenvalues $\mu^{(1)}, \ldots, \mu^{(r)}$.
Definition 2.11. For each ideal I_t and each m, k signed partial l-factor d define $V_L(I_t, d)$ to be the linear span of all $c \otimes d \otimes x$, where c is arbitrary and $x \in I_t$, i.e. x is a linear combination of elements for some $\sigma \in \tilde{S}_m$.

Note that $V_L(I_t, d)$ is a left ideal of $H_f^{(x)}(2k)$. Define $W_L(I_t, d) \subset V_L(I_t, d)$ to be the linear span of all $\sum (u)_{c,i}\, c \otimes d \otimes A_{i,t}$, where u is in $N^{[\lambda]}$ and where $A_{i,t}$ is the basis element of I_t corresponding to the basis element A_i in $S^{[\lambda]}$. i.e. the linear span of the set of all elements mapped to zero in $H_f^{(x)}(2k)$.
Proposition 2.12. Suppose $v = \sum (v)_{c,i}\, c \otimes d \otimes A_{i,t}$ is an element of $V_L(I_t, d)$. Let a, b be signed partial l-factors. For any $\sigma \in \tilde{S}_m$

$$(a \otimes b \otimes \sigma)v = a \otimes d \otimes \sigma \left\{\sum \gamma_j A_{j,t}\right\},$$

where γ_j is the (b, j) entry of $T_{m,k}(x)^{[\lambda]}(v)$.
Proof.

$$(a \otimes b \otimes \sigma) \circ \left\{\sum (v)_{c,i}\, c \otimes d \otimes A_{i,t}\right\} = a \otimes d \otimes \sigma \left\{\sum (v)_{c,i} < b, c > A_{i,t}\right\}$$
$$= a \otimes d \otimes \sigma \left\{\sum \gamma_j A_{j,t}\right\}$$

where γ_j is the coefficient of $A_{j,t}$ in $\sum (v)_{c,i} < b, c > A_{i,t}$. By definition of $T_{m,k}(x)^{[\lambda]}$, the coefficient of $A_{j,t}$ in $< b, c > A_{i,t}$ is the (b, j), (c, i) entry of $T_{m,k}(x)^{[\lambda]}$. Thus γ_j is the (b, j) entry of $T_{m,k}(x)^{[\lambda]}(v)$.
Proposition 2.13. (1) $H_f^{(x)}(2k) W_L(I_t, d) = 0$.

(2) $H(v) = V_L(I_t, d)$ for any v in $V_L(I_t, d)$ not in $W_L(I_t, d)$.

(3) $V_L(I_t, d)/W_L(I_t, d)$ is irreducible as a left $H_f^{(x)}(2k)$ module.

Proof: Suppose w is a generating element of $W_L(I_t, d)$. The γ_j appearing in Proposition 2.12 is all 0 for any $a \otimes b \otimes \sigma$ by the definition of $W_L(I_t, d)$. The first equation follows. Suppose v is in $V_L(I_t, d)$ but not in $W_L(I_t, d)$. Choose a (b, j) such that $\left(T_{m,k}(x)^{[\lambda]}(v)\right)$ is not zero. Then $a \otimes b \otimes \sigma(v)$ is not zero as γ_j is not zero. Note that a and σ were arbitrary. The images under $\sigma \in \hat{S}_m$ of any non-zero vector in I_t generate all of I_t as I_t is an irreducible $\hat{S}_m$ module. Hence vectors of the form

$$(a \otimes b \otimes \sigma) \circ \left(\sum (v)_{c,j}\, c \otimes d \otimes A_{j,t}\right)$$

generate all of $V_L(I_t, d)$. This proves the second inequality. The third follows immediately from the first two.
Let $W_L^{[\lambda]} = \oplus W_L(I_t, d)$. By Proposition 2.12, $W_L^{[\lambda]}$ is a nilpotent left ideal of $H_f^{(x)}(2k)$. Recall that $\hat{S}_m^{[\lambda]}$ can also be written as a direct sum of right ideals $J_1, \ldots, J_{d_{[\lambda]}}$. For each J_t and each signed partial l-factor a, let $V_R(J_t, a)$ be the linear span of all $a \otimes b \otimes x$, where b is arbitrary and x is in J_t. Define $W_R(J_t, a) \subset V_R(J_t, a)$ to be the linear span of all $\sum (u)_{c,i}\, a \otimes b \otimes A_{j,t}$ where $u^t T_{m,k}(x)^{[\lambda]} = 0$ and $A_{j,t}$ is as before.
The same proofs used in Propositions 2.12 and 2.13 show that

(1) $W_R(J_t, a) \circ (c \otimes d \otimes \sigma) = 0$,

(2) $V_R(J_t, a)/W_R(J_t, a)$ is an irreducible right $H_f^{(x)}(2k)$ module.

Define $W_R^{[\lambda]} = \oplus W_R(J_t, a)$ and define $W^{[\lambda]}$ to be the nilpotent 2-sided ideal $W^{[\lambda]} = W_L^{[\lambda]} + W_R^{[\lambda]}$.
Definition 2.14. Define $D^{[\lambda]}$ to be the 2-sided ideal of $H_f^{(x)}(2k)$ given by the linear span of all vectors $a \otimes b \otimes x$, where a and b are arbitrary and $x \in \hat{S}_m^{[\lambda]}$.

Note that $D^{[\lambda]}$ to be the 2-sided ideal of $H_f^{(x)}(2k)$. Note also that $H_f^{(x)}(2k)$ is the direct sum of the $D^{[\lambda]}$.
Proposition 2.15. $D^{[\lambda]}/W^{[\lambda]}$ is canonically isomorphic to the full matrix ring End($R^{[\lambda]}$). Recall that $R^{[\lambda]}$ is the range of $T_{m,k}(x)^{[\lambda]}$.
Proof: Given eigenvectors $v^{(r)}$ and $v^{(s)}$ define

$$Z(v^{(r)}, v^{(s)}) = (\mu^{(r)}\mu^{(s)})^{-1} \sum (v^{(r)})_{a,i} (v^{(s)})_{b,j}\, a \otimes b \otimes x_i y_j.$$

Taking the product of $Z(v^{(r)}, v^{(s)})$ and $Z(v^{(t)}, v^{(u)})$ we obtain

$$\begin{aligned} Z\left(v^{(r)}, v^{(s)}\right) Z\left(v^{(t)}, v^{(u)}\right) &= \left(\mu^{(r)}\mu^{(s)}\mu^{(t)}\mu^{(u)}\right)^{-1} \sum \left(v^{(r)}\right)_{a,i} (v^{(u)})_{d,l}\, a \otimes d \otimes \left(v^{(s)}\right)_{b,j} \left(v^{(t)}\right)_{c,k} \{ x_i y_j < b, c > x_k y_l \} \\ &= \left(\mu^{(r)}\mu^{(s)}\mu^{(t)}\mu^{(u)}\right)^{-1} \sum \left(v^{(r)}\right)_{a,i} (v^{(u)})_{d,l}\, a \otimes d \otimes x_i y_j \left(\sum (v^{(s)})_{b,j} \left\{ \sum (v^{(t)})_{c,k} < b, c > x_k \right\} \right) y_l \end{aligned}$$

Now $\sum (v^{(t)})_{c,k} < b, c > x_k = \sum \gamma_r x_r$, where γ_r is the b, r coefficient of $\left(T_{m,k}(x)^{[\lambda]}\right)(v^{(t)})$. In this case, $\gamma_r = \mu^{(t)} (v^{(t)})_{b,r}$ as $v^{(t)}$ is an eigenvector with eigenvalue $\mu^{(t)}$. So,

$$x_i y_j \left(\sum (v^{(s)})_{b,j} \left\{ \sum (v^{(t)})_{c,k} < b, c > x_k \right\} \right) y_l = \mu^{(t)} \sum \left(v^{(s)}\right)_{b,j} \left(v^{(t)}\right)_{b,r} \{ x_i y_j x_r y_l \}$$

But recall that

$$x_i y_j x_r y_l = \begin{cases} x_i y_l & \text{if } j = r \\ 0 & \text{otherwise} \end{cases}$$

Using this fact in the previous equation we have

$$x_i y_j \left(\sum (v^{(s)})_{b,j} \left\{ \sum (v^{(t)})_{c,k} < b, c > x_k \right\} \right) y_l = \mu^{(t)} x_i y_l \left\{ \sum \left(v^{(s)}\right)_{b,j} \left(v^{(t)}\right)_{b,j} \right\}.$$

By the orthonormality of the $v^{(i)}$ we have $\sum \left(v^{(s)}\right)_{b,j} \left(v^{(t)}\right)_{b,j} = \delta_{s,t}$, where $\delta_{s,t}$ is the Kronecker delta. Substituting above we obtain $Z\left(v^{(r)}, v^{(s)}\right) Z\left(v^{(t)}, v^{(u)}\right) = \delta_{s,t} Z\left(v^{(r)}, v^{(u)}\right)$, which shows that the subspace of $D^{[\lambda]}$ spanned by the $Z\left(v^{(r)}, v^{(s)}\right)$ is isomorphic to End($R^{[\lambda]}$).
The ideal $D^{[\lambda]} = V_{m,k} \otimes V_{m,k} \otimes \hat{S}_m^{[\lambda]}$ is isomorphic as a vector space to $\left(V_{m,k} \otimes S^{[\lambda]}\right) \otimes \left(V_{m,k} \otimes S^{[\lambda]}\right)$ via the linear map f sending $(c \otimes A_i) \otimes \left(d \otimes A_j\right)$ to $c \otimes d \otimes x_i y_j$. Writing $V_{m,k} \otimes S^{[\lambda]}$ as $N^{[\lambda]} \oplus R^{[\lambda]}$ we have, from Propositions 2.12, 2.13 and 2.15, that

(A) $f\left(N^{[\lambda]} \otimes \left(V_{m,k} \otimes S^{[\lambda]}\right) + \left(V_{m,k} \otimes S^{[\lambda]}\right) \otimes N^{[\lambda]}\right)$ is contained in the radical of $S_f^{(x)}(2k)$,

(B) $f\left(R^{[\lambda]} \otimes R^{[\lambda]}\right)$ is a full matrix ring.

The next theorem follows immediately from (A) and (B).
Theorem 2.16. With notation as above:

1. Let $W^{[\lambda]} = f\left(\mathrm{N} \otimes \left(V_{m,k} \otimes S^{[\lambda]}\right) + \left(V_{m,k} \otimes S^{[\lambda]}\right) \otimes \mathrm{N}\right)$. Then $W^{[\lambda]}$ is the intersection of the radical of $H_f^{(x)}(2k)$ with $D^{[\lambda]}$.
2. $D^{[\lambda]}/W^{[\lambda]}$ is a full matrix ring which is canonically isomorphic to End($R^{[\lambda]}$).

Conclusion

The irreducible representations of $S_f^{(x)}$ are indexed by partitions of m. Each such representations gives an irreducible representation of $S_f^{(x)}(2k)/S_f^{(x)}(2k+2)$. By the results in the above section, we get $Rad(S_f^{(x)}(2k))/Rad(S_f^{(x)}(2k+2)) = \sum W^{[\lambda]}$. This shows that the dimensions of the radical of the various $S_f^{(x)}(2k)$ are determined by the nullspaces of $T_{m,k'}(x)^{[\lambda]}$ for $k' \geq k$.

References

[1] R.Dipper and G.James, "Representations of Hecke algebras of type *Bn*," *J. Algebra* 146 (1992), no. 2, 454–481.

[2] P.Hanlon and D.Wales, "Eigenvalues connected with Brauer's centralizer algebras," *J. Algebra* 121(1989), no. 2, 446–476.

[3] G.James and A.Kerber, "The representation theory of the symmetric group," With a foreword by P. M. Cohn. With an introduction by Gilbert de B. Robinson. *Encyclopedia of Mathematics and its Applications*, 16.*Addison-Wesley Publishing Co., Reading, Mass.*, 1981.

[4] M.Parvathi and M.Kamaraj, "Signed Brauer's algebras," *Comm. Algebra,* 26 (1998), no. 3, 839–855.

[5] M.Parvathi and C.Selvaraj, "Signed Brauer's algebras as centralizer algebras," *Comm. Algebra,* 27 (1999), no. 12, 5985-5998.

[6] M.Parvathi and C.Selvaraj, "Characters of signed Brauer's algebras," *Southeast Asian Bull. Math.*30 (2006), no. 3, 495–514.