Convert An Analog Low Pass Filter To A Digital Filter Using Pascal's Triangle.

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ABSTRACT

This work introduces the relationship of the coefficients between a digital filter transfer function H(z) and an analog low pass filter transfer function H(s). By using the bilinear z-transformation pre-warping method with involving the Pascal's triangle, the formulas, are called Pascal matrix equation, is derived. The Pascal matrix equation is able to generate the coefficients of the digital filter from an analog prototype. The main key in this work is impressed the Pascal's triangle is involved to find the Pascal matrix equation of the relationship between digital and analog coefficients, and from that it can be computed and hand-calculated the conversion between analog low pass filter and digital filter.

Keywords : Bilinear z-transform, Pascal's triangle, low pass to low pass, low pass to high pass, low pass to band pass, low pass to band stop, matrix equation, pre-warping frequency.

1 INTRODUCTION

The bilinear z-transform is a popular method of one to one mapping the poles and zeros form a stable region of s-domain to a stable region of z-domain and vice versa. With the pre-warping frequency and frequency transform, the matrix equation of the relations between analog and digital coefficients is defined. In this paper, the involving of the Pascal's triangle in the matrix equation to use for conversion between a digital filter and an analog low pass filter is addressed.

2 THE PASCAL MATRIX EQUATION

The transfer function H(s) of an nth-ordered analog low pass filter in s-domain and the transfer function H(z) of a digital filter (low pass, high pass, band pass and band stop filter) in z-domain can be written as equation (1) below and where a_i , b_i and Ai, Bi are all real numbers, n is highest order number in the analog low pass filter and N= n for digital low pass and high pass filter and N=2n for digital band pass and band stop filter.

$$H(s)=\frac{\sum_{i=0}^{n}A_i s^i}{\sum_{i=0}^{n}A_i s^i} \qquad and \qquad H(z)=\frac{\sum_{i=0}^{N}a_i z^{-i}}{\sum_{i=0}^{N}b_i z^{-i}} \quad (1)$$

The common method to design a digital filter from a designed low pass filter is by first transform a designed low pass filter using frequency transformation to the same class of the desired digital filter, and then applying the bilinear z-transformation with pre-warping [2],[4]. And from that the matrix equation of the relationship between a_i , b_i and A_i , B_i are found as:

$$\begin{cases}[a]=[P][\Delta(A,U,L)]\\ [b]=[P][\Delta(B,U,L)]\end{cases} \quad (2)$$

If f_U is the upper cut off frequency of the band pass and band stop filter or the cut off frequency of the low pass filter, f_L is the lower cut off frequency of the band pass and band stop or the cut off frequency of the high pass filter and fs is the sampling frequency and then:

$$\begin{cases}c_U=\cot\left(\pi\frac{f_u}{f_s}\right)\\ t_L=\tan\left(\pi\frac{f_L}{f_s}\right)\end{cases}\Rightarrow\begin{cases}U=\frac{c_U}{1-c_U t_L}\\ L=\frac{t_L}{1-c_U t_L}\end{cases} \quad (3)$$

The matrices [a] and [b] in equation (2) have a size of (N+1 , 1) and they contain the digital coefficients in numerator and denominator of the transfer function H(z).

The matrix [P] contains the positive and negative binomial coefficients of the Pascal's triangle in the first, last row and the first, last column corresponding to the edge size and the nth row of the Pascal's triangle and another element in the matrix [P] can be calculated from its left, diagonal and above element. It has a size of (N+1 , N+1). There are two different matrices respectively for low pass filter P_{LP} and P_{HBS} for P_{HP} , P_{BP} , P_{BS} and they can be found as:

$Lp \rightarrow Lp$: P_{LP}

$$\left[(P_{LP})_{i=1\to n+1;\, j=1\to n+1}\right]_{n+1;\,n+1}=\left[\begin{cases}(P_{LP})_{i=1;\, j=1\to n+1}=1\\ (P_{LP})_{i=n+1;\, j=1\to n+1}=(-1)^{j-1}\\ (P_{LP})_{i=1\to n+1;\, j=1}=\binom{n}{i-1}\\ (P_{LP})_{i=1\to n+1;\, j=n+1}=(-1)^{i-1}\binom{n}{i-1}\\ (P_{LP})_{i,j}=(P_{LP})_{i,j-1}-(P_{LP})_{i-1,j-1}-(P_{LP})_{i-1,j}\end{cases}\right]_{n+1;\,n+1}$$

$Lp \rightarrow Hp$, Bp , Bs: P_{HP} , P_{BP} , P_{BS}

$$\left[(P_{HBS})_{i=1\to N+1;\, j=1\to N+1}\right]_{N+1;\,N+1}=\left[\begin{cases}(P_{HBS})_{i=1;\, j=1\to N+1}=1\\ (P_{HBS})_{i=N+1;\, j=1\to N+1}=(-1)^{j-1}\\ (P_{HBS})_{i=1\to N+1;\, j=1}=(-1)^{i-1}\binom{N}{i-1}\\ (P_{HBS})_{i=1\to N+1;\, j=N+1}=\binom{N}{i-1}\\ (P_{HBS})_{i,j}=(P_{HBS})_{i,j-1}+(P_{HBS})_{i-1,j-1}+(P_{HBS})_{i-1,j}\end{cases}\right]_{N+1;\,N+1}^{P_{HP}\to N=n;\; P_{BP},P_{BS}\to N=2n}$$

In this paper, the main key is impressed the involving of the Pascal's triangle of the expansion $(U+L)^n$ to find the matrix [Δ]. The Pascal's triangle with inserting zeros T is introduced in the fig.1 below.

n	$(U+L)^n$										
0	0	0	0	0	0	1	0	0	0	0	0
1	0	0	0	0	U	0	L	0	0	0	0
2	0	0	0	U^2	0	2UL	0	L^2	0	0	0
3	0	0	U^3	0	$3U^2L$	0	$3UL^2$	0	L^3	0	0
	0	...	0	...	0	...	0	...	0	...	0
n	U_n	0	...	0	...	...	...	0	...	0	L^n

Fig.1 The Pascal's triangle with insert zeros T

The matrix [Δ] in equation (2) has a size of (N+1, 1) and it can be found by multiplication between the coefficients matrix of the analog low pass filter and the matrix T of the Pascal's triangle with zeros as shown in equation (4). In table.1 below illustrated the matrix [Δ] for low pass, high pass, band pass and band stop filter.

$$\begin{cases}\left[(\Delta_{j-1}(A,U,L))_{i=1;j=1\to N+1}\right]_{1;N+1}=\left[(A_{i-1})_{i=1;j=1\to n+1}\right]_{1;n+1}\left[(T_{i;j})_{i=1\to n+1;j=1\to N+1}\right]_{n+1;N+1}\\ \left[(\Delta_{j-1}(B,U,L))_{i=1;j=1\to N+1}\right]_{1;N+1}B\left[(A_{i-1})_{i=1;j=1\to n+1}\right]_{1;n+1}\left[(T_{i;j})_{i=1\to n+1;j=1\to N+1}\right]_{n+1;N+1}\end{cases} \quad (4)$$

Table .1 The matrix [Δ] for Low pass, High pass, Band pass and Bans stop filt

n	[A] Lp,Hp, Bp	[A] Bs	The matrix [T]										
0	A_0	A_n	0	0	0	0	0	1	0	0	0	0	0
1	A_1	...	0	0	0	0	U	0	L	0	0	0	0
2	A_2	A_3	0	0	0	U^2	0	2UL	0	L^2	0	0	0
3	A_3	A_2	0	0	U^3	0	$3U^2L$	0	$3UL^2$	0	L^3	0	0
...	...	A_1	0	...	0	...	0	...	0	...	0	...	0
n	A_n	A_0	U^n	0	...	0	...	0	...	0	...	0	L^n
			$\Delta_{M=n}$	...	Δ_3	Δ_2	Δ_1	Δ_0	Lp to Lp N=n t=0 U=c				
Lp to Hp N=n c=0 L=t								Δ_0	Δ_1	Δ_2	Δ_3	...	$\Delta_{M=n}$
Lp to Bp , Bs N=2n			Δ_0	Δ_1	Δ_2	...	Δ_{M-1}	$\Delta_{M=n}$	Δ_{M+1}	Δ_{M+2}	...	Δ_{2n-1}	Δ_{2n}

Base on equation (2) and equation (4), one new formula can derived and it is called "The Pascal Matrix Equation" as is described in equation (5)

$$\begin{cases}\left[(a_{i-1})_{i=1\to N+1;j=1}\right]_{N+1;1}=\left[(P)_{i=1\to N+1;j=1\to N+1}\right]_{N+1;N+1}\left(\left[(A)_{i=1;j=1\to N+1}\right]_{1;n+1}\left[(T_{i;j})_{i=1\to n+1;j=1\to N+1}\right]_{n+1;N+1}\right)_{N+1;1}\\ \left[(b_{i-1})_{i=1\to N+1;j=1}\right]_{N+1;1}=\left[(P)_{i=1\to N+1;j=1\to N+1}\right]_{N+1;N+1}\left(\left[(B)_{i=1;j=1\to N+1}\right]_{1;n+1}\left[(T_{i;j})_{i=1\to n+1;j=1\to N+1}\right]_{n+1;N+1}\right)_{N+1;1}\end{cases} \quad (5)$$

3 Converse An Analog Low Pass Filter To A Digital Filter

The Pascal matrix equation is a general formula to transform the coefficients of an analog filter transfer function H(s) to the coefficients of a digital filter transfer function H(z) and it is described more detail in the sections of the converting low pass to low pass, high pass, band pass and band stop.

3.1 Converse an analog low pass filter to a digital low pass filter

Form table.1, the matrix [P] is the matrix [PLP]. The matrix [Ai], [Bi] coefficients in the numerator and denominator of the transfer function H(s) of the analog low pass filter are written in one row, n+1 columns and the subscripts is in ascending order. Let t=0, then L=0 and U=c, the matrix [T] is then rewritten as a diagonal matrix [Tc] with the power c of n and has a size of (n+1 ; n+1) as below:

$$[T_c]_{n+1;n+1}=\begin{bmatrix}1&0&0&0&0\\0&c&0&0&0\\0&0&c^2&0&0\\0&0&0&\ddots&0\\0&0&0&0&c^n\end{bmatrix}_{n+1;\,n+1}$$

The Pascal matrix equation of low pass to low pass can be expressed as below:

$$\begin{cases}\left[(a_{i-1})_{i=1\to N+1;j=1}\right]_{N+1;1}=\left[(P)_{i=1\to N+1;j=1\to N+1}\right]_{N+1;N+1}\left(\left[(A_{j-1})_{i=1;j=1\to N+1}\right]_{1;n+1}\left[(T_c)_{i=1\to n+1;j=1\to N+1}\right]_{n+1;N+1}\right)_{N+1;1}\\ \left[(b_{i-1})_{i=1\to N+1;j=1}\right]_{N+1;1}=\left[(P)_{i=1\to N+1;j=1\to N+1}\right]_{N+1;N+1}\left(\left[(B_{j-1})_{i=1;j=1\to N+1}\right]_{1;n+1}\left[(T_c)_{i=1\to n+1;j=1\to N+1}\right]_{n+1;N+1}\right)_{N+1;1}\end{cases} \quad (6)$$

Example 1: Convert a 4th-order elliptic low pass filter has a transfer function H(s) with 3 dB of ripple in the pass band, and 20 dB of attenuation in the stop band to a digital low pass filter has the cut off frequency at 400 Hz and sampling frequency of 2 kHz.

$$H(s)=\frac{0.3405+0.4158s^2+0.1s^4}{0.481+0.514s+1.4943s^2+0.5463s^3+s^4}$$

$$n=4,t=0,c=\cot(\pi\frac{400}{2000})=1.3764$$

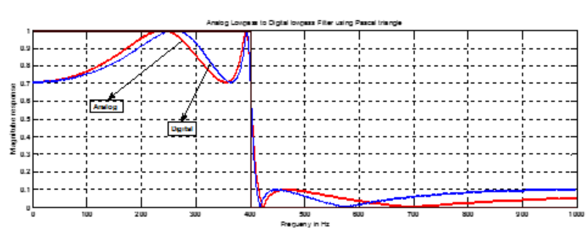
$$\begin{cases}[A_i]_{1;5}=[0.3405\quad 0\quad 0.4158\quad 0\quad 0.1]\\ [B_i]_{1;5}=[0.4810\quad 0.5140\quad 1.4943\quad 0.5463\quad 1]\end{cases}$$

From equation (6), the matrix [ai] and [bi] can be calculated as

$$\begin{cases}[a_i]_{5;1}=\begin{bmatrix}1&1&1&1&1\\4&2&0&-2&-4\\6&0&-2&0&6\\4&-2&0&2&-4\\1&-1&1&-1&1\end{bmatrix}_{5;5}\left([0.3405\quad 0\quad 0.4158\quad 0\quad 0.1]_{1;5}\begin{bmatrix}1&0&0&0&0\\0&1.3764&0&0&0\\0&0&1.8944&0&0\\0&0&0&2.6075&0\\0&0&0&0&3.5889\end{bmatrix}_{5;5}\right)_{5;1}\\ [b_i]_{5;1}=\begin{bmatrix}1&1&1&1&1\\4&2&0&-2&-4\\6&0&-2&0&6\\4&-2&0&2&-4\\1&-1&1&-1&1\end{bmatrix}_{5;5}\left([0.4810\quad 0.5140\quad 1.4943\quad 0.5463\quad 1]_{1;5}\begin{bmatrix}1&0&0&0&0\\0&1.3764&0&0&0\\0&0&1.8944&0&0\\0&0&0&2.6075&0\\0&0&0&0&3.5889\end{bmatrix}_{5;5}\right)_{5;1}\end{cases}$$

$$\begin{cases}[a_i]_{5;1}=\begin{bmatrix}1&1&1&1&1\\4&2&0&-2&-4\\6&0&-2&0&6\\4&-2&0&2&-4\\1&-1&1&-1&1\end{bmatrix}\begin{bmatrix}0.3405\\0\\0.7877\\0\\0.3589\end{bmatrix}=\begin{bmatrix}1.4871\\-0.0735\\2.6210\\-0.0735\\1.4871\end{bmatrix}\\ [b_i]_{5;1}=\begin{bmatrix}1&1&1&1&1\\4&2&0&-2&-4\\6&0&-2&0&6\\4&-2&0&2&-4\\1&-1&1&-1&1\end{bmatrix}\begin{bmatrix}0.4810\\0.7075\\2.8308\\1.4245\\3.5889\end{bmatrix}=\begin{bmatrix}9.0327\\-13.8656\\18.7578\\-10.9977\\4.7687\end{bmatrix}\end{cases}$$

$$\therefore H(z)=\frac{1.4871-0.0735z^{-1}+2.6210z^{-2}-0.0735z^{-3}+1.4871z^{-4}}{9.0327-13.8656z^{-1}+18.7578z^{-2}-10.9977z^{-3}+4.7687z^{-4}}$$



3.2 Converse an analog low pass filter to a digital high pass filter

Converting a low pass filter to a digital high pass filter is similar manner, the matrix [P] is the $[P_{BHS}]$ with the matrix's size is (n+1;n+1) and the matrix [T] is $[T_t]$ which is replaced c by t.

$$[T_t]_{n+1;n+1}=\begin{bmatrix}1&0&0&0&0\\0&t&0&0&0\\0&0&t^2&0&0\\0&0&0&\ddots&0\\0&0&0&0&t^n\end{bmatrix}_{n+1;\,n+1}$$

The Pascal matrix equation of low pass to high pass can be written as

$$\begin{cases}\left[(a_{i-1})_{i=1\to N+1;j=1}\right]_{N+1;1}=\left[(P)_{i=1\to N+1;j=1\to N+1}\right]_{N+1;N+1}\left(\left[\left[(A_{j-1})_{i=1;j=1\to N+1}\right]_{1;n+1}\left[(T_i)_{i=1\to n+1;j=1\to N+1}\right]_{n+1;N+1}\right]\right)_{N+1;1}\\ \left[(b_{i-1})_{i=1\to N+1;j=1}\right]_{N+1;1}=\left[(P)_{i=1\to N+1;j=1\to N+1}\right]_{N+1;N+1}\left(\left[\left[(B_{j-1})_{i=1;j=1\to N+1}\right]_{1;n+1}\left[(T_i)_{i=1\to n+1;j=1\to N+1}\right]_{n+1;N+1}\right]\right)_{N+1;1}\end{cases}\quad(7)$$

Example 2: Convert a 4th-order elliptic low pass filter has a transfer function H(s) with 3 dB of ripple in the pass band, and 20 dB of attenuation in the stop band to a digital high pass filter has the cut off frequency at 400 Hz and sampling frequency of 2 kHz.

$$H(s)=\frac{0.3405+0.4158s^2+0.1s^4}{0.481+0.514s+1.4943s^2+0.5463s^3+s^4}$$

$$n=4, c=0, t=\tan(\pi\frac{400}{2000})=0.7265$$

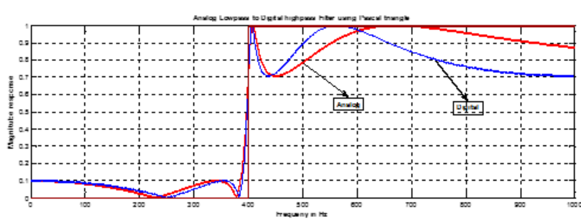
$$\begin{cases}[A_i]_{1;5}=[0.3405\quad 0\quad 0.4158\quad 0\quad 0.1]\\ [B_i]_{1;5}=[0.4810\quad 0.5140\quad 1.4943\quad 0.5463\quad 1]\end{cases}$$

From equation (7), the matrix [a$_i$] and [b$_i$] can be calculated as below

$$\begin{cases}[a_i]_{5;1}=\begin{bmatrix}1&1&1&1&1\\-4&-2&0&2&4\\6&0&-2&0&6\\-4&2&0&-2&4\\1&-1&1&-1&1\end{bmatrix}_{5;5}\left([0.3405\quad 0\quad 0.4158\quad 0\quad 0.1]_{1;5}\begin{bmatrix}1&0&0&0&0\\0&0.7265&0&0&0\\0&0&0.5279&0&0\\0&0&0&0.3835&0\\0&0&0&0&0.2786\end{bmatrix}_{5;5}\right)_{5;1}\\ [b_i]_{5;1}=\begin{bmatrix}1&1&1&1&1\\-4&-2&0&2&4\\6&0&-2&0&6\\-4&2&0&-2&4\\1&-1&1&-1&1\end{bmatrix}_{5;5}\left([0.4810\quad 0.5140\quad 1.4943\quad 0.5463\quad 1]_{1;5}\begin{bmatrix}1&0&0&0&0\\0&0.7265&0&0&0\\0&0&0.5279&0&0\\0&0&0&0.3835&0\\0&0&0&0&0.2786\end{bmatrix}_{5;5}\right)_{5;1}\end{cases}$$

$$\begin{cases}[a_i]_{5;1}=\begin{bmatrix}1&1&1&1&1\\-4&-2&0&2&4\\6&0&-2&0&6\\-4&2&0&-2&4\\1&-1&1&-1&1\end{bmatrix}\begin{bmatrix}0.3405\\0\\0.2195\\0\\0.0279\end{bmatrix}=\begin{bmatrix}0.5879\\-1.2506\\1.7713\\-1.2506\\0.5879\end{bmatrix}\\ [b_i]_{5;1}=\begin{bmatrix}1&1&1&1&1\\-4&-2&0&2&4\\6&0&-2&0&6\\-4&2&0&-2&4\\1&-1&1&-1&1\end{bmatrix}\begin{bmatrix}0.4810\\0.3735\\0.7888\\0.2095\\0.2786\end{bmatrix}=\begin{bmatrix}2.1314\\-1.1373\\2.9802\\-0.4815\\0.9654\end{bmatrix}\end{cases}$$

$$\therefore H(z)=\frac{0.5879-1.2506z^{-1}+1.7713z^{-2}-1.2506z^{-3}+0.5879z^{-4}}{2.1314-1.1373z^{-1}+2.9802z^{-2}-0.4815z^{-3}+0.9654z^{-4}}$$



The formulas (equation (6) and (7)) of converting of the analog low pass filter to a digital low pass and high pass filter are presented and demonstrated. The matrix [T] can be found from the edges of the Pascal's triangle. In the next sections, it is more interesting for converting an analog low pass filter to a digital band pass and band stop with the Pascal's triangle.

3.3 Converse an analog low pass filter to a digital band pass filter

This section shows how to obtain the transfer function H(z) of the digital band pass from the analog low pass transfer function H(s). Due to the low pas to Band pass frequency transformation, the order of the band pass filter becomes N=2n, where n is the order of the analog low pass filter. Therefore, the matrix [P_{HBS}] has a size of (2n+1; 2n+1) and the matrix [Ai], [Bi] are the same as in converting low pass to low pass and high pass. From table.1, the matrix [T] is shown as below

$$[T]=\begin{bmatrix}0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&U&0&L&0&0&0&0\\0&0&0&U^2&0&2UL&0&L^2&0&0&0\\0&0&U^3&0&3U^2L&0&3UL^2&0&L^3&0&0\\0&\cdot^{\cdot^{\cdot}}&0&\cdots&0&\cdots&0&\cdots&0&\ddots&0\\U^n&0&\cdots&0&\cdots&0&\cdots&0&\cdots&0&L^n\end{bmatrix}$$

From equation (5), the Pascal matrix equation of converting an analog low pass filter to a digital band pass filter can be written as

$$\begin{cases}\left[(a_{i-1})_{i=1\to N+1;j=1}\right]_{N+1;1}=\left[(P_{HBS})_{i=1\to N+1;j=1\to N+1}\right]_{N+1;N+1}\left(\left[\left[(A_{j-1})_{i=1;j=1\to N+1}\right]_{1;n+1}\left[(T)_{i=1\to n+1;j=1\to N+1}\right]_{n+1;N+1}\right]\right)_{N+1;1}\\ \left[(b_{i-1})_{i=1\to N+1;j=1}\right]_{N+1;1}=\left[(P_{HBS})_{i=1\to N+1;j=1\to N+1}\right]_{N+1;N+1}\left(\left[\left[(B_{j-1})_{i=1;j=1\to N+1}\right]_{1;n+1}\left[(T)_{i=1\to n+1;j=1\to N+1}\right]_{n+1;N+1}\right]\right)_{N+1;1}\end{cases}\quad(8)$$

Example 3: Convert a 4th-order elliptic low pass filter has a transfer function H(s) with 3 dB of ripple in the pass band, and 20 dB of attenuation in the stop band to a digital band pass filter has the cut off frequency at 400 Hz and sampling frequency of 2 kHz.

$$H(s)=\frac{0.3405+0.4158s^2+0.1s^4}{0.481+0.514s+1.4943s^2+0.5463s^3+s^4}$$

$$n=4, c=\cot(\pi\frac{1000}{10000})=0.7265, t=\tan(\pi\frac{3000}{10000})=0.3249$$

$$U=\frac{c}{1-ct}=0.9511\ ;\ L=\frac{t}{1-ct}=0.4253$$

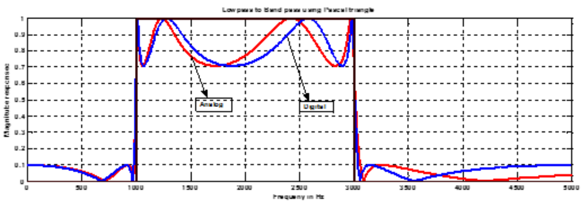
$$\begin{cases}[A_i]_{1;5}=[0.3405\quad 0\quad 0.4158\quad 0\quad 0.1]\\ [B_i]_{1;5}=[0.4810\quad 0.5140\quad 1.4943\quad 0.5463\quad 1]\end{cases}$$

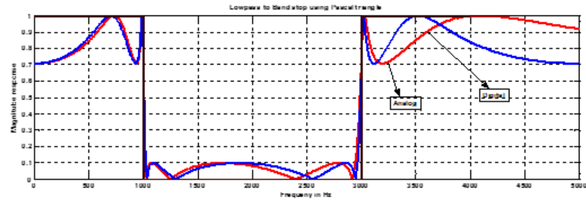
From equation (8), the matrix [a$_i$] and [bi] can be calculated as below:

$$\begin{cases}([A]_{1;5}[T]_{5;9})_{9;1}=\left([0.3405\ 0\ 0.4158\ 0\ 0.1]\begin{bmatrix}0&0&0&0&1&0&0&0&0\\0&0&0&0.9511&0&0.4253&0&0&0\\0&0&0.9045&0&0.8090&0&0.1809&0&0\\0&0.8602&0&1.1541&0&0.5161&0&0.0769&0\\0.8181&0&1.4635&0&0.9818&0&0.2927&0&0.0327\end{bmatrix}\right)_{9;1}=\begin{bmatrix}0.0818\\0\\0.5225\\0\\0.7751\\0\\0.1045\\0\\0.0033\end{bmatrix}\\ ([B]_{1;5}[T]_{5;9})_{9;1}=\left([0.4810\ 0.5140\ 1.4943\ 0.5463\ 1]\begin{bmatrix}0&0&0&0&1&0&0&0&0\\0&0&0&0.9511&0&0.4253&0&0&0\\0&0&0.9045&0&0.8090&0&0.1809&0&0\\0&0.8602&0&1.1541&0&0.5161&0&0.0769&0\\0.8181&0&1.4635&0&0.9818&0&0.2927&0&0.0327\end{bmatrix}\right)_{9;1}=\begin{bmatrix}0.8181\\0.4700\\2.8151\\1.1194\\2.6717\\0.5006\\0.5630\\0.0420\\0.0327\end{bmatrix}\end{cases}$$

$$\begin{cases}[a]_{9;1}=\begin{bmatrix}1&1&1&1&1&1&1&1&1\\-8&-6&-4&-2&0&2&4&6&8\\28&14&4&-2&-4&-2&4&14&28\\-56&-14&4&6&0&-6&-4&14&56\\70&0&-10&0&6&0&-10&0&70\\-56&14&4&-6&0&6&-4&-14&56\\28&-14&4&2&-4&2&4&-14&28\\-8&6&-4&2&0&-2&4&-6&8\\1&-1&1&-1&1&-1&1&-1&1\end{bmatrix}\begin{bmatrix}0.0818\\0\\0.5225\\0\\0.7751\\0\\0.1045\\0\\0.0033\end{bmatrix}=\begin{bmatrix}1.4871\\-2.3002\\1.7898\\-2.7264\\4.3371\\-2.7264\\1.7898\\-2.3002\\1.4871\end{bmatrix}\\ [b]_{9;1}=\begin{bmatrix}1&1&1&1&1&1&1&1&1\\-8&-6&-4&-2&0&2&4&6&8\\28&14&4&-2&-4&-2&4&14&28\\-56&-14&4&6&0&-6&-4&14&56\\70&0&-10&0&6&0&-10&0&70\\-56&14&4&-6&0&6&-4&-14&56\\28&-14&4&2&-4&2&4&-14&28\\-8&6&-4&2&0&-2&4&-6&8\\1&-1&1&-1&1&-1&1&-1&1\end{bmatrix}\begin{bmatrix}0.8181\\0.4700\\2.8151\\1.1194\\2.6717\\0.5006\\0.5630\\0.0420\\0.0327\end{bmatrix}=\begin{bmatrix}9.0326\\-19.0967\\30.5779\\-37.2528\\41.8088\\-32.6964\\22.7222\\-11.4866\\4.7687\end{bmatrix}\end{cases}$$

$$\therefore H(z)=\frac{1.4871-2.3002z^{-1}+1.7898z^{-2}-2.7264z^{-3}+4.3371z^{-4}-2.7264z^{-5}+1.7898z^{-6}-2.3002z^{-7}+1.4871z^{-8}}{9.0326-19.0967z^{-1}+30.5779z^{-2}-37.2528z^{-3}+41.8088z^{-4}-32.6964z^{-5}+22.7222z^{-6}-11.4866z^{-7}+4.7687z^{-8}}$$





3.4 Converse an analog low pass filter to a digital band stop filter

Converting a low pass filter to a digital band stop filter is the same way as converting the low pass filter to a digital band pass filter. Only one different is the subscript of the coefficients in the matrix [Ai] and [Bi] are written in descending order. The Pascal matrix equation can be expressed as below:

$$\begin{cases} \left[(a_{i-1})_{i=1\to N+1;j=1}\right]_{N+1;1} = \left[(P_{HBS})_{i=1\to N+1;j=1\to N+1}\right]_{N+1;N+1}\left(\left[(A_{n+1-j})_{i=1;j=1\to N+1}\right]_{1;n+1}\left[(T)_{i=1\to n+1;j=1\to N+1}\right]_{n+1;N+1}\right)_{N+1;1} \\ \left[(b_{i-1})_{i=1\to N+1;j=1}\right]_{N+1;1} = \left[(P_{HBS})_{i=1\to N+1;j=1\to N+1}\right]_{N+1;N+1}\left(\left[(B_{n+1-j})_{i=1;j=1\to N+1}\right]_{1;n+1}\left[(T)_{i=1\to n+1;j=1\to N+1}\right]_{n+1;N+1}\right)_{N+1;1} \end{cases} \quad (9)$$

Example 4: Convert a 4th-order elliptic low pass filter has a transfer function H(s) with 3 dB of ripple in the pass band, and 20 dB of attenuation in the stop band to a digital band stop filter has the cut off frequency at 400 Hz and sampling frequency of 2 kHz.

$$H(s) = \frac{0.3405 + 0.4158s^2 + 0.1s^4}{0.481 + 0.514s + 1.4943s^2 + 0.5463s^3 + s^4}$$

$$n = 4, c = \cot(\pi\frac{1000}{10000}) = 0.7265, t = \tan(\pi\frac{3000}{10000}) = 0.3249$$

$$U = \frac{c}{1-ct} = 0.9511 \; ; \; L = \frac{t}{1-ct} = 0.4253$$

$$\begin{cases} [A_i]_{1;5} = [0.3405 \quad 0 \quad 0.4158 \quad 0 \quad 0.1] \\ [B_i]_{1;5} = [0.4810 \quad 0.5140 \quad 1.4943 \quad 0.5463 \quad 1] \end{cases}$$

$$\begin{cases} ([A]_{1;5}[T]_{5;9})_{9;1} = \left([0.1 \; 0 \; 0.4158 \; 0 \; 0.3405]\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9511 & 0 & 0.4253 & 0 & 0 & 0 \\ 0 & 0 & 0.9045 & 0 & 0.8090 & 0 & 0.1809 & 0 & 0 \\ 0 & 0.8602 & 0 & 1.1541 & 0 & 0.5161 & 0 & 0.0769 & 0 \\ 0.8181 & 0 & 1.4635 & 0 & 0.9818 & 0 & 0.2927 & 0 & 0.0327 \end{bmatrix}\right)_{9;1} = \begin{bmatrix} 0.2786 \\ 0 \\ 0.8745 \\ 0 \\ 0.7707 \\ 0 \\ 0.1749 \\ 0 \\ 0.0111 \end{bmatrix} \\ ([B]_{1;5}[T]_{5;9})_{9;1} = \left([1 \; 0.5463 \; 1.4943 \; 0.5140 \; 0.4810]\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9511 & 0 & 0.4253 & 0 & 0 & 0 \\ 0 & 0 & 0.9045 & 0 & 0.8090 & 0 & 0.1809 & 0 & 0 \\ 0 & 0.8602 & 0 & 1.1541 & 0 & 0.5161 & 0 & 0.0769 & 0 \\ 0.8181 & 0 & 1.4635 & 0 & 0.9818 & 0 & 0.2927 & 0 & 0.0327 \end{bmatrix}\right)_{9;1} = \begin{bmatrix} 0.3935 \\ 0.4422 \\ 2.0555 \\ 1.1128 \\ 2.6811 \\ 0.4977 \\ 0.4111 \\ 0.0396 \\ 0.0157 \end{bmatrix} \end{cases}$$

$$\begin{cases} [a]_{9;1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -8 & -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 \\ 28 & 14 & 4 & -2 & -4 & -2 & 4 & 14 & 28 \\ -56 & -14 & 4 & 6 & 0 & -6 & -4 & 14 & 56 \\ 70 & 0 & -10 & 0 & 6 & 0 & -10 & 0 & 70 \\ -56 & 14 & 4 & -6 & 0 & 6 & -4 & -14 & 56 \\ 28 & -14 & 4 & 2 & -4 & 2 & 4 & -14 & 28 \\ -8 & 6 & -4 & 2 & 0 & -2 & 4 & -6 & 8 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0.2786 \\ 0 \\ 0.8745 \\ 0 \\ 0.7707 \\ 0 \\ 0.1749 \\ 0 \\ 0.0111 \end{bmatrix} = \begin{bmatrix} 2.1098 \\ -4.9378 \\ 9.2271 \\ -12.1787 \\ 14.4120 \\ -12.1787 \\ 9.2271 \\ -4.9378 \\ 2.1098 \end{bmatrix} \\ [b]_{9;1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -8 & -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 \\ 28 & 14 & 4 & -2 & -4 & -2 & 4 & 14 & 28 \\ -56 & -14 & 4 & 6 & 0 & -6 & -4 & 14 & 56 \\ 70 & 0 & -10 & 0 & 6 & 0 & -10 & 0 & 70 \\ -56 & 14 & 4 & -6 & 0 & 6 & -4 & -14 & 56 \\ 28 & -14 & 4 & 2 & -4 & 2 & 4 & -14 & 28 \\ -8 & 6 & -4 & 2 & 0 & -2 & 4 & -6 & 8 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0.3935 \\ 0.4422 \\ 2.0555 \\ 1.1128 \\ 2.6811 \\ 0.4977 \\ 0.4111 \\ 0.0396 \\ 0.0157 \end{bmatrix} = \begin{bmatrix} 7.6493 \\ -13.2461 \\ 14.1247 \\ -16.5238 \\ 20.0683 \\ -12.6317 \\ 7.0779 \\ -5.9538 \\ 3.4648 \end{bmatrix} \end{cases}$$

The procedure of converting an analog low pass filter to a digital filter were studied, the involving of the Pascal's triangle in the Pascal matrix equation made the work easier for compute and hand-calculation.

4 Conclusion

The new method were studied for converse from an analog low pass filter with the transfer function H(s) to a digital filter (low pass, high pass, band pass and band stop) with the transfer function H(z). The involving of the Pascal's triangle is used in the Pascal matrix equation made the work easier for hand-calculation and computing when transforming between s-domain and z-domain. The algorithm of this method conversion is so simple due to all operations imply the matrix multiplication and so it is easy to program and calculation.

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