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Convergence of Three-Step Iterations Scheme for Non-self Asymptotically Non-expansive Mappings

Abstract : The aim of this paper is to prove the weak and strong convergence of the three-step iterative sequence for non-self asymptotically non-expansive mappings in a real uniformly convex Banach space. The results presented in this paper improve and generalize some recent papers by Suantai [7], Khan and Hussain [10], Nilsrakoo and Saejung [6], and many others.

Key-words : Banach space, retract, non-expansive non-self-mapping, uniformly convex Banach, asymptotically non-expansive mapping, non-self asymptotically non-expansive mappings, non-self I-asymptotically quasi non expansive mapping

Introduction : Suppose that X is a real uniformly convex Banach space, K is a non-empty closed convex subset of X. Let T be a self-mapping of K. A mapping T is called non-expansive provided $\| Tx - Ty \| \leq \| x - y \|$, for all $x, y \in K$. T is called asymptotically non-expansive mapping if there exists a sequence

$\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\| T^n x - T^n y \| \leq k_n \| x - y \|$

for all $x, y \in K$ and $n \geq 1$.

Let X be a real normed space and K be a non-empty subset of X. A subset K of X is called retract of X if there exists a continuous map $P : X \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P : X \to K$ is called a retraction if $P^2 = P$. In particular, a subset K is called a non-expansive retract of X if there exists a non-expansive retraction $P : X \to K$ such that $Px = x$ for all $x \in K$.

Iterative techniques for converging fixed points of non-expansive non-self-mappings have been studied by many authors (see, e.g., Khan and Hussain [10], Wang [11]).Suantai [7] introduced iterative process and used it for the weak and strong convergence of fixed points of self-mappings in a uniformly convex Banach space.

The concept of non-self asymptotically non-expansive mappings was introduced by Chidume [12] as the generalization of asymptotically non-expansive self-mappings and obtained some strong and weak convergence theorems for mappings given as follows :

for $x_1 \in K$,

$$y_n = P\,(\beta_n T\,(PT)^{n-1}\,x_n + (1-\beta_n)\,x_n),$$
$$x_{n+1} = P\,(\alpha_n T\,(PT)^{n-1}\,y_n + (1-\alpha_n)\,x_n),\ \ \forall\ n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$.

A non-self mapping T is called asymptotically non-expansive if there exists a Sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|\ T\,(PT)^{n-1}\,x - T\,(PT)^{n-1}\,y\ \| \leq k_n\ \|\ x - y\ \|$$

for all $x, y \in K$, and $n \geq 1$.

T is called uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|\ T\,(PT)^{n-1}\,x - T\,(PT)^{n-1}\,y\ \| \leq L\ \|\ x - y\ \|$$

for all $x, y \in K$, and $n \geq 1$. From the above definition, it is obvious that non-self asymptotically non-expansive mappings are uniformly L-Lipschitzian.

Now, I give the following non-self-version for $x_1 \in K$,

$$z_n = P\,(a_n T\,(PT)^{n-1}\,x_n + (1-a_n)\,x_n),$$
$$y_n = P\,(b_n T\,(PT)^{n-1}\,z_n + c_n T\,(PT)^{n-1}\,x_n + (1-b_n-c_n)\,x_n),$$
$$x_{n+1} = P\,(\alpha_n T\,(PT)^{n-1}\,y_n + \beta_n T\,(PT)^{n-1}\,z_n + \gamma_n T\,(PT)^{n-1}\,z_n$$
$$+ \gamma_n T\,(PT)^{n-1}\,x_n + (1-\alpha_n-\beta_n-\gamma_n)\,x_n),$$

for all $n \geq 1$, where $\{a_n\}$, $\{c_n\}$, $\{b_n + c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\alpha_n + \beta_n + \gamma_n\}$ in $[0, 1]$ satisfy certain conditions.

Lemma 2.1 (see [12]). Let X be a uniformly convex Banach space, K a non-empty closed convex subset of X and $T : K \to K$ a non-self asymptotically non-expansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$, then $I - T$ is demiclosed at zero.

Lemma 2.2 (see [12]). Let X be a real uniformly convex Banach space, K a non-empty closed subset of X with P as a sunny non-expansive retraction and $T : K \to X$ a mapping satisfying weakly inward condition, then $F\,(PT) = F\,(T)$.

Lemma 2.3 (see [14]). Let $\{s_n\}$, $\{t_n\}$, and $\{\sigma_n\}$ be sequences of nonnegative real sequences satisfying the following conditions : for all $n \geq 1$, $s_{n+1} \leq (1 + \sigma_n)\,s_n + t_n$, where $\sigma_n < \infty$ and $t_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.

Lemma 2.4 (233 [6]). Let X be a uniformly convex Banach space and $B_R := \{x \in X : \|x\| \leq R\}$, $R > 0$, then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g\,(0) = 0$ such that

$$\|\ \lambda x + \mu y + \xi z + v w\ \|^2 \leq\ \lambda\ \|\ x\ \|^2 + \mu\ \|\ y\ \|^2 + \xi\ \|\ z\ \|^2 + v\ \|\ w\ \|^2 -\ v\,(\lambda g\,(\|\ x - w\ \|)$$
$$+ \mu g\,(\|\ y - w\ \|) + \xi g\,(\|\ z - w\ \|),$$

for all $x, y, z, w \in B_r$ and $\lambda, \mu, \xi, v \in [0, 1]$ with $\lambda + \mu + \xi + v = 1$.

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Lemma 2.5 (See [7], Lemma 2.7). Let X be a Banach space which satisfies Opial's condition and let x_n be a sequence in X. Let $q_1, q_2 \in X$ be such that $\lim_{n\to\infty} \|x_n - q_1\|$ and $\lim_{n\to\infty} \| x_n - q_2 \|$. If $\{x_{nk}\}$, $\{x_{nj}\}$ are the subsequences of $\{x_n\}$ which converge weakly to $q_1, q_2 \in X$, respectively, then

Main results :

Theorem 3.2. Let X be a real Banach space and K a non-empty closed and convex subset of X. Let $T : K \to X$ be a non-self I-asymptotically quasi non expansive mapping with sequence $\{k_n\} \subset [1, \infty)$; $(k_n - 1) < \infty$; $I : K \to X$ be a non-self asymptotically non-expansive mapping with $\{l_n\} \subset [1, \infty)$; $(l_n - 1) < \infty$; P be a retraction from X to K. Let $\{a_n\}$ $\{b_n$ $\{c_n\}$. $\{\alpha_n\}$ $\{\beta_n\}$ $\{\gamma_n\}$ be real sequence in $[0, 1]$ such that $\{b_n + c_n\}$, $\{\alpha_n + \beta_n + \gamma_n\}$ in $[0, 1]$; $\forall\ n \geq 1$, $\{x_n\}$ be defined as

$$z_n = P\,(a_n\, T\,(PT)^{n-1})\, x_n + (1 - a_n)\, x_n)$$

$$y_n = P\,(b_n\, T\,(PT)^{n-1}\, z_n + c_n\, T\,(PT)^{n-1}\, x_n + (1 - b_n - c_n)$$

$$x_{n+1} = P\,(\alpha_n\, I\,(PI)^{n-1}\, y_n + \beta_n\, I\,(PI)^{n-1}\, z_n + \gamma_n\, I\,(PI)^{n-1}\, x_n + (1 - \alpha_n - \beta_n - \gamma_n)\, x_n)$$

then for any $q \in F(T) \cap F(I)$, $\lim_{n\to\infty} \| x_n - q \|$ exists.

Proof : Considering

$$\begin{aligned}
\| z_n - q \| &= \| P\,(a_n\, T\,(PT)^{n-1}\, x_n + (1 - a_n)\, x_n) - Pq \| \\
&\leq \| a_n\, (T\,(PT)^{n-1}\, x_n - q) + (1 - a_n)\, (x_n - q) \| \\
&\leq a_n \| T\,(PT)^{n-1}\, x_n - q \| + (1 - a_n) \| x_n - q \| \\
&\leq a_n k_n \| I\,(PI)^{n-1}\, x_n - q \| + (1 - a_n) \| x_n - q \| \\
&\leq a_n\, k_n\, l_n \| x_n - q \| + (1 - a_n) \| x_n - q \| \\
&= [1 + a_n\, (k_n l_n - 1)] \| x_n - q \|.
\end{aligned}$$

$$\begin{aligned}
\| y_n - q \| &= \| P\,(b_n T\,(PT)^{n-1}\, z_n + c_n T\,(PT)^{n-1}\, x_n + (1 - b_n - c_n)\, x_n - Pq \| \\
&\leq b_n \| T\,(PT)^{n-1}\, z_n - q \| + c_n \| T\,(PT)^{n-1}\, x_n - q \| + (1 - b_n - c_n) \| x_n - q \| \\
&\leq b_n\, k_n \| I\,(PI)^{n-1}\, z_n - q \| + c_n\, k_n \| x_n - q \| + (1 - b_n - c_n) \| x_n - q \| \\
&\leq b_n\, k_n\, l_n \| z_n - q \| + c_n\, k_n\, l_n \| x_n - q \| + (1 - b_n - c_n) \| x_n - q \| \\
&\leq b_n\, k_n\, l_n\, [1 + a_n\, (k_n\, l_n - 1)] \| x_n - q \| + (c_n k_n l_n + 1 - b_n - c_n) \| x_n - q \| \\
&= [1 + (k_n l_n - 1)\, (b_n + c_n + a_n b_n k_n l_n)] \| x_n - q \|
\end{aligned}$$

$$\| x_{n+1} - q \| = \| P\,(\alpha_n I\,(PI)^{n-1} y_n + \beta_n\, I\,(PI)^{n-1}\, z_n + \gamma_n\, I\,(PI)^{n-1} x_n$$

$$+ (1 - \alpha_n - \beta_n - \gamma_n)\, x_n) - Pq \,\|$$

$$\leq \alpha_n l_n \| y_n - q \| + \beta_n l_n \| z_n - q \| + \gamma_n l_n \| x_n - q \|$$

$$+ (1 - \alpha_n - \beta_n - \gamma_n) \| x_n - q \|$$

$$\leq \alpha_n l_n [1 + (k_n l_n - 1)(b_n + c_n + a_n b_n k_n l_n)] \| x_n - q \|$$

$$+ \beta_n l_n [1 + a_n (k_n l_n - 1)] \| x_n - q \| + \gamma_n l_n \| x_n - q \|$$

$$+ (1 - \alpha_n - \beta_n - \gamma_n) \| x_n - q \|$$

$$\leq [1 + (l_n - 1)(\alpha_n + \beta_n + \gamma_n) + (k_n l_n - 1)[\alpha_n b_n + \alpha_n c_n]$$

$$+ (k_n l_n - 1)[a_n b_n k_n \alpha_n l_n^2] + (k_n l_n - 1)(\beta_n l_n a_n)] \| x_n - q \|$$

Hence, $\lim_{n\to\infty} \| x_n - q \|$ exists.

Theorem 3.4. Let X be a real uniformly convex Banach space and K a non-empty closed convex subset of X. Let $T : K \to X$ be a non-self asymptotically non-expansive mapping with the non-empty fixed-point set $F(T)$ and a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $< \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real sequences in [0, 1], such that $\{b_n + c_n\}$ nd $\{\alpha_n + \beta_n + \gamma_n\}$ in [0, 1] for all $n \geq 1$. Let $\{x_n\}$ be a sequence in K defined by (1) with the following restrictions :

(1) $0 < \min\{\liminf_n \alpha_n, \liminf_n \beta_n, \liminf_n \gamma_n\} \leq \limsup_n (\alpha_n + \beta_n + \gamma_n) < 1$ and $\limsup_n a_n < 1$,

(2) $0 \leq \limsup_n b_n \leq \limsup_n (b_n + c_n) < 1$,

then $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$.

Proof : We first consider

$$\| x_{n+1} - x_n \| \leq \| (\alpha_n T (PT)^{n-1} y_n + \beta_n T (PT)^{n-1} z_n + \gamma_n T (PT)^{n-1} x_n$$

$$+ (1 - \alpha_n - \beta_n - \gamma_n)\, x_n) - x_n \|$$

$$\leq \alpha_n \| T (PT)^{n-1} y_n - x_n \| + \beta_n \| T (PT)^{n-1} z_n - x_n \|$$

$$+ \gamma_n \| T(PT)^{n-1} x_n - x_n \| \to 0, \text{ as } n \to \infty.$$

We note that every asymptotically non-expansive mapping is uniformly L-Lipschitzian. Also note that

$$\| x_{n+1} - T (PT)^{n-1} x_{n+1} \| \leq \| x_{n+1} - x_n \| + \| x_n - T (PT)^{n-1} x_n \|$$

$$+ \| T (PT)^{n-1} x_n - T (PT)^{n-1} x_{n+1} \|$$

$$= \| x_{n+1} - x_n \| + \| T (PT)^{n-1} x_{n+1} - T (PT)^{n-1} x_n \|$$

$$+ \| T (PT)^{n-1} x_n - x_n \|$$

$$\leq \| x_{n+1} - x_n \| + L \| x_{n+1} - x_n \| + \| T (PT)^{n-1} x_n - x_n \|$$

$$\to 0, \text{ as } n \to \infty.$$

In addition,

$$\| x_{n+1} - T(PT)^{n-2} x_{n+1} \| \le \| x_{n+1} - x_n \| + \| x_n - T(PT)^{n-2} x_n \| + \| T(PT)^{n-2} x_n - T(PT)^{n-2} x_{n+1} \|$$
$$\le \| x_{n+1} - x_n \| + \| T(PT)^{n-2} x_n - x_n \| + L \| x_{n+1} - x_n \|$$
$$\to 0, \text{ as } n \to \infty.$$

We denote as $(PT)^{1-1}$ the identity maps from K into itself. Thus, by above inequality, we write

$$\| x_{n+1} - Tx_{n+1} \| \le \| x_{n+1} - T(PT)^{n-1} x_{n+1} \| + \| T(PT)^{n-1} x_{n+1} - Tx_{n+1} \|$$
$$= \| x_{n+1} - T(PT)^{n-1} x_{n+1} \| + L \| T(PT)^{n-2} x_{n+1} - x_{n+1} \| \to 0,$$

as $n \to \infty$.

which implites that

$$\lim_{n\to\infty} \| x_n - Tx_n \| = 0.$$

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