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# Concerning Fuzzy Strongly Irresolute Multifunctions

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## ABSTRACT

In this paper some types of fuzzy multifunctions have been introduced and studied and find mutual relationships among themselves.

**Keywords:** *Fuzzy strongly upper (lower) irresolute multifunctions, fuzzy strongly upper (lower) semi-continuous multifunctions.*

## 1 INTRODUCTION

IN1985, Papageorgiou [5] introduced fuzzy multifunctions, a function from an ordinary topological space $X$ to a fuzzy topological space $Y$ and from then a group of researchers are engaged themselves for studying different types of fuzzy multifunctions. Papagerogiou defined upper and lower inverses of a fuzzy multifunction. Afterwards, in 1991 Mukherjee and Malakar [4] suitably redefined lower inverse and it seems that this new definition of lower inverse is more useful than the previous one. In this paper the definition of new lower inverse given by Mukherjee and Malakar and the definition of upper inverse given by Papageorgiou have been used.

In what follows, by $(X, \tau)$ or simply $X$ we shall mean an ordinary topological space, while $(Y, \tau_1)$ or simply $Y$ stands for a fuzzy topological space (fts, for short) in the sense of Chang [1]. $cl\, A$ and $int\, A$ of a set $A$ in $X$ (respectively, a fuzzy set [7] in $Y$) respectively stand for the closure and interior of $A$ in $X$ (respectively, in $Y$). $0_Y$ and $1_Y$ are the constant fuzzy sets taking respectively the constant values 0 and 1 on $Y$. The complement of a fuzzy set $A$ in $Y$ will be denoted by $1_Y - A$ [7], defined by $(1_Y - A)(y) = 1 - A(y)$, for all $y \in Y$. For two fuzzy sets $A$ and $B$ in $Y$, we write $A \leq B$ will mean $A(y) \leq B(y)$, for all $y \in Y$, whereas we write $A\, q\, B$ to mean $A$ is quasi-coincident (q-coincident, for short) with $B$ [6] if there is some $y \in Y$ such that $A(y) + B(y) > 1$; the negation of $A\, q\, B$ is written as $A\, \bar{q}\, B$. A set $A$ in a topological space $X$ is said to be semiopen [3] if there exists an open set $U$ in $X$ such that $U \subseteq A \subseteq cl\, int\, A$. The set of all semiopen sets in $X$ will be denoted by $SO(X)$. The complement of a semiopen set in $X$ is called a semiclosed set. The semiclosure of a set $A$ in $X$, to be written as $scl\, A$, is the set of all points $x$ in $X$ such that for every semiopen set $U$ in $X$ with $x \in U$, it follows that $U \cap A \neq \emptyset$ [2]. A set $A$ in $X$ is semiclosed iff $A = scl\, A$. The union of all semiopen sets in $X$ contained in a set $A$ is called the semi-interior of $A$, denoted by $sint\, A$. It is clear that a set $A$ is semiopen iff $A = sint\, A$.

## 2 FUZZY STRONGLY IRRESOLUTE MULTIFUNCTIONS

Let us recall the following definitions from [5, 4].

### Definition 2.1.

Let $(X, \tau)$ and $(Y, \tau_1)$ be respectively an ordinary topological space and an fts. We say that $F : X \to Y$ is a fuzzy multifunction if corresponding to each $x \in X$, $F(x)$ is a unique fuzzy set in $Y$.
Henceforth by $F : X \to Y$ we shall mean a fuzzy multifunction in the above sense.

### Definition 2.2.

For a fuzzy multifunction $F : X \to Y$, the upper inverse $F^+$and the lower inverse $F^-$ are defined as follows :
For any fuzzy set $A$ in $Y$, $F^+(A) = \{x \in X : F(x) \leq A\}$ and $F^-(A) = \{x \in X : F(x) q\, A\}$.

The following theorem shows a mutual relationship between the upper and the lower inverses of a fuzzy multifunction.

### Theorem 2.3. [4]

For a fuzzy multifunction $: X \to Y$ , we have $F^-(1_Y \setminus A) = X \setminus F^+(A)$, for any fuzzy set $A$ in $Y$.

### Definition 2.4.

A fuzzy multifunction $F : X \to Y$ is said to be fuzzy strongly irresolute at a point $x$ of $X$ if for any two fuzzy sets $B_1$ , $B_2$ of $Y$ such that $F(x) \leq B_1$ and $F(x) q\, B_2$, there exists $U \in SO(X)$ containing $x$ such that $F(U) \leq B_1$ and $F(u) q\, B_2$, for all $u \in U$.
$F : X \to Y$ is called fuzzy strongly irresolute on $X$ if $F$ is fuzzy strongly irresolute at every point $x$ of $X$.

### Theorem 2.5.

The following are equivalent for a fuzzy multifunction $: X \to Y$ :

(a) $F$ is fuzzy strongly irresolute at a point $x \in X$,
(b) For any two fuzzy sets $B_1$ , $B_2$ of $Y$ such that $F(x) \leq B_1$ and $F(x) q\, B_2$, $x \in cl\, (int\, (F^+(B_1) \cap F^-(B_2)))$,
(c) For any two fuzzy sets $B_1$ , $B_2$ of $Y$ such that $F(x) \leq B_1$ and $F(x) q\, B_2$ and for any open set $U$ of $X$ containing $x$, there exists an open set $G_U$ $(\neq \emptyset)$ of $X$ such that $G_U \subset U$, $F(G_U) \leq B_1$, $F(g) q\, B_2$ for every $g \in G_U$.

 

**Proof.**
(a)⇒ (b): Let $F : X \to Y$ be fuzzy strongly irresolute at a point $x \in X$ and $B_1, B_2$ be any two fuzzy sets in $Y$ such that $F(x) \leq B_1$ and $F(x)q\, B_2$. Then by (a), there exists $U \in SO(X)$ such that $F(U) \leq B_1$ and $F(u)q\, B_2$, for all $u \in U$. Therefore, $x \in U \subseteq (F^+(B_1) \cap F^-(B_2))$ and hence $x \in U \subseteq cl\,(int\, U) \subseteq cl\,(int\,(F^+(B_1) \cap F^-(B_2)))$.

(b) ⇒ (c): Let $B_1, B_2$ be any two fuzzy sets in $Y$ such that $F(x) \leq B_1$ and $F(x)q\, B_2$ and $U$ be any open set in $X$ containing $x$. Then $x \in cl\,(int\,(F^+(B_1) \cap F^-(B_2)))$, and hence $U \cap int\,(F^+(B_1) \cap F^-(B_2)) \neq \emptyset$. Put $G_U = U \cap int\,(F^+(B_1) \cap F^-(B_2))$. Then $\emptyset \neq G_U \subset U$, $G_U \subset F^+(B_1)$ and $G_U \subset F^-(B_2)$. Then there exists a non-empty open set $G_U$ such that $G_U \subset U, F(U) \leq B_1$ and $F(g)q\, B_2$, for all $g \in G_U$.

(c) ⇒ (a): Let $\mathcal{U}(x)$ be any family of all open sets in $X$ containing $x$. Let $B_1, B_2$ be any two fuzzy sets in $Y$ such that $F(x) \leq B_1$ and $F(x)q\, B_2$. For each $U \in \mathcal{U}(x)$, by (c), there exists a non-empty open set $G_U$ of $X$ such that $G_U \subset U$, $F(G_U) \leq B_1$, $F(g)q\, B_2$ for every $g \in G_U$. Let $W = \cup\{G_U : U \in \mathcal{U}(x)\}$. Then $W$ is open in $X$, $x \in cl\, W$, $F(W) \leq B_1$ and $F(w)q\, B_2$, for all $w \in W$. Now put $S = W \cup \{x\}$. Then $W \subset S \subset cl\, W$ and hence $S \in SO(X)$ containing $x$. Moreover, $F(S) \leq B_1$ and $F(S)q\, B_2$, for all $s \in S$. Hence $F$ is fuzzy strongly irresolute multifunction.

**Theorem 2.6.**
For a fuzzy multifunction $F : X \to Y$, the following are equivalent :

(a) $F$ is fuzzy strongly irresolute on $X$,
(b) For any two fuzzy sets $B_1$, $B_2$ of $Y$, $F^+(B_1) \cap F^-(B_2) \in SO(X)$,
(c) For any two fuzzy sets $B_1$, $B_2$ of $Y$, $F^+(B_1) \cup F^-(B_2)$ is semiclosed in $X$,
(d) For any two fuzzy sets $B_1$, $B_2$ of $Y$, $scl\,(F^+(B_1) \cup F^-(B_2)) \subset F^+(B_1) \cup F^-(B_2)$,
(e) For any two fuzzy sets $B_1$, $B_2$ of $Y$, $sint\,(F^+(B_1) \cap F^-(B_2)) \supset F^+(B_1) \cap F^-(B_2)$.

**Proof.**
(a)⇒(b): Let $B_1$, $B_2$ be any two fuzzy sets in $Y$ and $x \in F^+(B_1) \cap F^-(B_2)$. Then $F(x) \leq B_1$ and $F(x)q\, B_2$. Then by Theorem 2.5 (a) ⇒ (b), $x \in cl\,(int\,(F^+(B_1) \cap F^-(B_2)))$ and hence $int\,(F^+(B_1) \cap F^-(B_2)) \subset F^+(B_1) \cap F^-(B_2) \subset cl\,(int\,(F^+(B_1) \cap F^-(B_2)))$. Therefore, $F^+(B_1) \cap F^-(B_2) \in SO(X)$.

(b)⇒(c): Let $B_1$, $B_2$ be any two fuzzy sets in $Y$. Then by (b), we have $X\backslash(F^+(B_1) \cup F^-(B_2))= (X\backslash F^+(B_1)) \cap (X\backslash F^-(B_2)) = F^-(1_Y\backslash B_1) \cap F^+(1_Y\backslash B_2) \in SO(X)$. Therefore, $F^+(B_1) \cup F^-(B_2)$ is semiclosed in $X$.

(c)⇒(d): Obvious.

(d)⇒(e): For any two fuzzy sets $B_1$, $B_2$ of $Y$, $X\backslash sint\,(F^+(B_1) \cap F^-(B_2)) = scl\,((X\backslash F^+(B_1) \cup (X\backslash F^-(B_2))) = scl\,(F^-(1_Y\backslash B_1) \cup F^+(1_Y\backslash B_2)) \subset (F^-(1_Y\backslash B_1) \cup F^+(1_Y\backslash B_2)) = (X\backslash F^+(B_1) \cup (X\backslash F^-(B_2)) = X\backslash(F^+(B_1) \cap F^-(B_2))$. Therefore, $sint\,(F^+(B_1) \cap F^-(B_2)) \supset F^+(B_1) \cap F^-(B_2)$.

(e)⇒(a): Let $x$ be any point of $X$. Let $B_1$, $B_2$ be any two fuzzy sets in $Y$ such that $F(x) \leq B_1$ … (i) and $F(x)q\, B_2$…. (ii).
By (e), $sint\,(F^+(B_1) \cap F^-(B_2)) \supset F^+(B_1) \cap F^-(B_2)$ and put $U = F^+(B_1) \cap F^-(B_2)$. Then $x \in U$ (from (i) and (ii)) and $U \in SO(X)$ containing $x$, $F(U) \leq B_1$ and $F(u)q\, B_2$ for every $u \in U$. Therefore, $F$ is fuzzy strongly irresolute.

**Theorem 2.7.**
Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of a space $X$. A fuzzy multifunction $F : X \to Y$ is fuzzy strongly irresolute iff the restriction $F_\alpha = F/U_\alpha : U_\alpha \to Y$ is fuzzy strongly irresolute for each $\alpha \in \Lambda$.

**Proof.**
Let $\alpha$ be an arbitrary fixed index of $\Lambda$. Let $x \in U_\alpha$ and $B_1$, $B_2$ be any two fuzzy sets in $Y$ such that $F_\alpha(x) \leq B_1$ and $F_\alpha(x)q\, B_2$. Since $F_\alpha(x) = F(x)$ and $F$ is fuzzy strongly irresolute, there exists $U \in SO(X)$ containing $x$ such that $F(U) \leq B_1$ and $F(u)q\, B_2$ for all $u \in U$. Let $G_\alpha = U \cap U_\alpha$. Then $G_\alpha \in SO(U_\alpha)$ containing $x$, $F_\alpha(G_\alpha) = F(G_\alpha) \leq B_1$ and $F_\alpha(u)q\, B_2$, for all $u \in G_\alpha$. Therefore, $F_\alpha$ is fuzzy strongly irresolute.
Conversely, suppose that $F_\alpha$ is fuzzy strongly irresolute for each $\alpha \in \Lambda$. Let $x \in X$ and $B_1$, $B_2$ be any two fuzzy sets in $Y$ such that $F(x) \leq B_1$ and $F(x)q\, B_2$. Then there exists $\alpha \in \Lambda$ such that $x \in U_\alpha$. Since $F(x) = F_\alpha(x)$, we have $F_\alpha(x) \leq B_1$ and $F_\alpha(x)q\, B_2$. By definition, there exists $G_\alpha \in SO(U_\alpha)$ containing $x$ such that $F_\alpha(G_\alpha) \leq B_1$ and $F_\alpha(u)q\, B_2$, for all $u \in G_\alpha$. It follows that $G_\alpha \in SO(X)$ containing $x$ and hence $F$is fuzzy strongly irresolute.

## 3 FUZZY STRONGLY UPPER (LOWER) SEMI-CONTINUOUS MULTIFUNCTIONS

In this section, two different types of fuzzy multifunctions have been introduced and characterized them and given a mutual relationships between themselves.

**Definition 3.1.**
A fuzzy multifunction $F : X \to Y$ is said to be fuzzy strongly upper (lower) semi-continuous if for any fuzzy set $B$ of $Y$, $F^+(B)$ (resp. $F^-(B)$) is open in $X$.

**Theorem 3.2.**
For a fuzzy multifunction $: X \to Y$ , the following are equivalent:

(a) $F$ is fuzzy strongly lower semi-continuous.
(b) For each $x \in X$ and any fuzzy set $B$ of $Y$ with $F(x)q\, B$, there exists an open set $U$ of $X$ containing $x$ such that $F(u)q\, B$, for each $u \in U$.
(c) For each fuzzy set $B$ of $Y$, $F^+(B)$ is closed in $X$.
(d) For each fuzzy set $B$ of $Y$, $cl\,(F^+(B)) \subseteq F^+(B)$.
(e) For each subset $A$ of $X$, $F(cl\, A) \leq F(A)$.

**Proof.**
(a)⟹(b): Let $x \in X$ and $B$ be any fuzzy set of $Y$ with $F(x)q\,B$. Then $x \in F^-(B)$. Put $U = F^-(B)$. Then by (a) $U$ is an open set in $X$ containing $x$. Now $F(U) = FF^-(B) = \{F(x): x \in F^-(B)\} = \{F(x): F(x)q\,B\}$. Therefore, $F(u)q\,B$ for each $u \in U$.

(b)⟹(c): Let $B$ be any fuzzy set of $Y$. Let $x \in X\backslash F^+(B) = F^-(1_Y\backslash B)$. Then $F(x)q\,(1_Y\backslash B)$. Then by (b), there exists an open set $U$ in $X$ containing $x$ such that $F(u)q\,(1_Y\backslash B)$ for each $\in U$. Therefore, $U \subseteq F^-(1_Y\backslash B) = X\backslash F^+(B)$. Therefore, $x \in U \subseteq X\backslash F^+(B)$ where $U$ is an open set in $X$ which shows that $X\backslash F^+(B)$ is open in $X$ and hence $F^+(B)$ is closed in $X$.
(c)⟹(d): Obvious.

(d)⟹(e): Let $A$ be any subset of $X$. Then $cl\,A \subseteq cl\left(F^+\left(F(A)\right)\right) \subseteq F^+(F(A))$. Therefore, $F(cl\,A) \leq F(A)$.

(e)⟹(b): Let $x \in X$ and $B$ be any fuzzy set of $Y$ such that $F(x)q\,B$. Then $x \in F^-(B)$. By (e), $F\left(cl\left(F^+(1_Y\backslash B)\right)\right) \leq F\left(F^+(1_Y\backslash B)\right) \leq 1_Y\backslash B \Rightarrow cl\left(F^+(1_Y\backslash B)\right) \subseteq F^+(1_Y\backslash B) \Rightarrow F^+(1_Y\backslash B) = X\backslash F^-(B)$ is closed in $X \Rightarrow F^-(B)$ is open in $X$. Let $U = F^-(B)$. Then $x \in U$ and $F(u)q\,B$ for each $u \in U$.

(b)⟹(a): Let $B$ be any fuzzy set of $Y$. Let $x \in F^-(B)$. Then $F(x)q\,B$. Then by (b), there exists an open set $U$ in $X$ containing $x$ such that $F(u)q\,B$ for each $u \in U \Rightarrow U \subseteq F^-(B)$. Therefore, $x \in U \subseteq F^-(B)$ where $U$ is open in $X$.
This completes the proof.

**Theorem 3.3.**
For a fuzzy multifunction $: X \to Y$ , the following are equivalent:
(a) $F$ is fuzzy strongly upper semi-continuous.
(b) For each point $x \in X$ and any fuzzy set $B$ of $Y$ with $F(x) \leq B$, there exists an open set $U$ in $X$ containing $x$ such that $F(U) \leq B$.
(c) For each fuzzy set $B$ of $Y$, $F^-(B)$ is closed in $X$.
(d) For each fuzzy set $B$ of $Y$, $cl\left(F^-(B)\right) \subseteq F^-(B)$.
(e) For each fuzzy set $B$ of $Y$, $int\left(F^+(B)\right) \supseteq F^+(B)$.

**Proof.**
(a)⟹(b): Let $x \in X$ and $B$ be any fuzzy set of $Y$ with $F(x) \leq B$. Then $x \in F^+(B)$. Let $U = F^+(B)$. Then by (a), $U$ is an open set in $X$ containing $x$. Now $F(U) = FF^+(B) = \{F(x): x \in F^+(B)\} = \{F(x): F(x) \leq B\} \Longrightarrow F(U) \leq B$.

(b)⟹(c): Let $B$ be any fuzzy set of $Y$. Let $x \in X\backslash F^-(B) = F^+(1_Y\backslash B)$. Then $F(x) \leq (1_Y\backslash B)$. Then by (b), there exists an open set $U$ in $X$ containing $x$ such that $F(U) \leq (1_Y\backslash B) \Rightarrow U \subseteq F^+(1_Y\backslash B) = X\backslash F^-(B) \Longrightarrow x \in U \subseteq X\backslash F^-(B)$ which shows that $X\backslash F^-(B)$ is open in $X$ and hence $F^-(B)$ is closed in $X$.

(c)⟹(d): Obvious.

(d)⟹(e): Let $B$ be any fuzzy set of $Y$. Then $X\backslash F^+(B) = F^-(1_Y\backslash B) \supseteq cl\,(F^-(1_Y\backslash B)) = cl\left(X\backslash F^+(B)\right) = X\backslash intF^+(B)$. Therefore, $int\left(F^+(B)\right) \supseteq F^+(B)$.

(e)⟹(a): Let $B$ be any fuzzy set of $Y$. Then by (e), $F^+(B) \subseteq intF^+(B) \Rightarrow F^+(B)$ is open in $X$. Hence $F$ is fuzzy strongly upper semi-continuous.
This completes the proof.

**Definition 3.4.**
A fuzzy multifunction $F: X \to Y$ is said to be
(a) Fuzzy strongly upper irresolute at a point $x \in X$ if for any fuzzy set $B$ of $Y$ such that $F(x) \leq B$, there exists $U \in SO(X)$ containing $x$ such that $F(U) \leq B$,
(b) Fuzzy strongly lower irresolute at a point $x \in X$ if for any fuzzy set $B$ of $Y$ such that $F(x)\,q\,B$, there exists $U \in SO(X)$ containing $x$ such that $F(u)\,q\,B$, for each $u \in U$.

$F: X \to Y$ is called fuzzy strongly upper (lower) irresolute on $X$ if $F$ is fuzzy strongly upper (lower) irresolute at every point $x \in X$.

**Theorem 3.5.**
For a fuzzy multifunction $: X \to Y$ , the following are equivalent :
(a) $F$ is fuzzy strongly lower irresolute at a point $x \in X$,
(b) $x \in cl(int(F^-(B)))$ for every fuzzy set $B$ of $Y$ with $F(x)q\,B$,
(c) For any open set $U$ of $X$ containing $x$ and for any fuzzy set $B$ of $Y$ with $F(x)q\,B$, there exists a non-empty open set $G \subset U$ such that $F(g)q\,B$, for each $g \in G$.

**Proof.**
(a)⟹(b) : Let $B$ be any fuzzy set in $Y$ with $F(x)q\,B$. Then $x \in F^-(B)$. Then by (a), there exists $U \in SO(X)$ containing $x$ such that $F(u)q\,B$, for each $u \in U$. Then $x \in U \subseteq F^-(B)$ and hence $x \in U \subseteq cl\left(int(U)\right) \subseteq cl\left(int\left(F^-(B)\right)\right)$.

(b)⟹(c) : Let $B$ be any fuzzy set in $Y$ with $F(x)q\,B$ and $U$ be an open set in $X$ containing $x$. Then $x \in cl(int(F^-(B)))$ and hence $U \cap int(F^-(B)) \neq \emptyset$. Let $G = U \cap int(F^-(B))$. Then $\emptyset \neq G \subset U$, $G \subset F^-(B)$. Therefore, there exists a non-empty open set $G$ such that $G \subset U, F(g)q\,B$, for each $g \in G$.

(c)⟹(a) : Let $\mathcal{U}(x)$ be any family of all open sets in $X$ containing $x$. Let $B$ be any fuzzy set in $Y$ with $F(x)q\,B$. For each $U \in \mathcal{U}(x)$, by (c), there exists a non-empty open set $G_U$ of $X$ such that $G_U \subset U$, $F(g)q\,B$, for every $g \in G_U$. Let $W = \cup\{G_U : U \in \mathcal{U}(x)\}$. Then $W$ is open in $X$, $x \in cl\,W$, $F(w)q\,B$, for all $w \in W$. Now put $S = W \cup \{x\}$. Then $W \subset S \subset cl\,W$ and hence $S \in SO(X)$ containing $x$. Moreover, $F(S)q\,B$, for all $s \in S$. Hence $F$ is fuzzy strongly lower irresolute multifunction at $x$.

**Theorem 3.6.**
For a fuzzy multifunction $: X \to Y$ , the following are equivalent :
(a) $F$ is fuzzy strongly lower irresolute.
(b) For each fuzzy set $B$ of $Y$, $F^-(B) \in SO(X)$.

(c) For each point $x \in X$ and any fuzzy set $B$ of $Y$ with $F(x)q\,B$, there exists $U \in SO(X)$ containing $x$ such that $F(u)q\,B$, for each $u \in U$.
(d) For each fuzzy set $B$ of $Y$, $F^{+}(B)$ is semi-closed in $X$.
(e) For each fuzzy set $B$ of $Y$, $int\left(cl\left(F^{+}(B)\right)\right) \subset F^{+}(B)$.
(f) For each subset $A$ of $X$, $F(int\,(cl\,A))) \le F(A)$.
(g) For each subset $A$ of $X$, $F(scl\,A) \le F(A)$.

**Proof.**
(a)⇒(b) : Let $B$ be any fuzzy set of $Y$ and $x \in F^{-}(B)$. Then $F(x)q\,B$. Then by Theorem 3.5 (a) ⇒ (b), $x \in cl(int(F^{-}(B)))$. Then we have, $F^{-}(B) \subset cl(int(F^{-}(B)))$. Therefore, $F^{-}(B) \in SO(X)$.

(b) ⇒ (c) : Let $x \in X$ and $B$ be any fuzzy set of $Y$ such that $F(x)q\,B$. Then $x \in F^{-}(B)$. Put $U = F^{-}(B)$. Then $U \in SO(X)$ containing $x$ and $F(u)q\,B$, for each $u \in U$. In fact $F(U) = FF^{-}(B) = \{F(x): x \in F^{-}(B)\} = \{F(x): F(x)q\,B\}$.
(c) ⇒ (d) : Let $B$ be any fuzzy set of $Y$. Let $x \in X\backslash F^{+}(B) = F^{-}(1_Y\backslash B)$. Then $F(x)q\,(1_Y\backslash B)$. By (c), there exists $U \in SO(X)$ containing $x$ such that $F(u)q\,(1_Y\backslash B)$, for each $u \in U$. Therefore, $u \in F^{-}(1_Y\backslash B)$ for each $u \in U \Rightarrow U \subset F^{-}(1_Y\backslash B) = X\backslash F^{+}(B)$ and hence $x \in U \subseteq cl\left(int(U)\right) \subseteq cl\,(int(X\backslash F^{+}(B)))$. Hence $int\left(X\backslash F^{+}(B)\right) \subset X\backslash F^{+}(B) \subset cl\,(int(X\backslash F^{+}(B)))$. Therefore, $X\backslash F^{+}(B)$ is semiopen in $X$ and hence $F^{+}(B)$ is semiclosed in .

(d) ⇒ (e) : Let $B$ be any fuzzy set of $Y$. Then $F^{+}(B)$ is semiclosed in $X$ and hence $int\left(cl\left(F^{+}(B)\right)\right) \subset F^{+}(B)$.

(e) ⇒ (f) : Let $A$ be any subset of $X$. Then $int(cl\,A) \subset int\left(cl\left(F^{+}(F(A))\right)\right) \subset F^{+}(F(A))$ (by (e)) and so $F(int\,(cl\,A))) \le F(A)$.

(f) ⇒ (g) : Let $A$ be any subset of $X$. Then $scl\,A = A \cup int(cl\,A)$ and hence $F(scl\,A) = F(A) \cup F\left(int(cl\,A)\right) \le F(A) \cup F(A)$ (by (f)) $= F(A)$.

(g) ⇒ (a) : Let $x \in X$ and $B$ be any fuzzy set of $Y$ such that $F(x)q\,B$. Then $x \in F^{-}(B)$. Then by (g), $F\left(scl\left(F^{+}(1_Y\backslash B)\right)\right) \le F\left(F^{+}(1_Y\backslash B)\right) \le 1_Y\backslash B$. Therefore, $scl\left(F^{+}(1_Y\backslash B)\right) \le F^{+}(1_Y\backslash B)$. Therefore, $F^{+}(1_Y\backslash B) = scl\,(F^{+}(1_Y\backslash B))$ and so $F^{+}(1_Y\backslash B) = X\backslash F^{-}(B)$ is semiclosed in $X$ and hence $F^{-}(B)$ is semiopen in $X$. Put $U = F^{-}(B)$. Then $U \in SO(X)$ containing $x$ and $F(u)q\,B$, for each $u \in U$. Therefore, $F$ is fuzzy strongly lower irresolute.

**Theorem 3.7.**
For a fuzzy multifunction $F: X \longrightarrow Y$, the following are equivalent :
(a) $F$ is fuzzy strongly upper irresolute at a point $x \in X$.
(b) $x \in \mathrm{cl}\,(\mathrm{int}(F^{+}(B)))$ for any fuzzy set $B$ of $Y$ with $F(x) \le B$.
(c) For any open set $U$ of $X$ containing $x$ and for any fuzzy set $B$ of $Y$ with $F(x) \le B$, there exists a non-empty open set $G \subset U$ such that $F(G) \le B$.

**Proof.**
(a) ⇒ (b) : Let $B$ be any fuzzy set of $Y$ with $F(x) \le B$. Then $x \in F^{+}(B)$. Then by (a), there exists $U \in SO(X)$ containing $x$ such that $F(U) \le B$. Then $x \in U \subseteq cl\,(intU) \subseteq cl\,(int(F^{+}(B)))$.

(b) ⇒ (c) : Let $B$ be any fuzzy set of $Y$ with $F(x) \le B$ and $U$ be any open set in $X$ containing $x$. Then by (b), $x \in \mathrm{cl}\,(\mathrm{int}(F^{+}(B)))$ and hence $U \cap int(F^{+}(B)) \neq \emptyset$. Let $G = U \cap int(F^{+}(B))$. Then $\emptyset \neq G \subset U$, $G \subset F^{+}(B)$. Therefore, there exists a non-empty open set $G$ such that $G \subset U, F(G) \le B$.

(c) ⇒ (a): Let $\mathcal{U}(x)$ be any family of all open sets in $X$ containing $x$. Let $B$ be any fuzzy set in $Y$ such that $F(x) \le B$. For each $U \in \mathcal{U}(x)$, by (c), there exists a non-empty open set $G_U$ of $X$ such that $G_U \subset U$, $F(G_U) \le B$. Let $W = \cup\{G_U : U \in \mathcal{U}(x)\}$. Then $W$ is open in $X$, $x \in cl\,W$, $F(W) \le B$ . Now put $S = W \cup \{x\}$. Then $W \subset S \subset cl\,W$ and hence $S \in SO(X)$ containing $x$. Moreover, $F(S) \le B$ . Hence $F$ is fuzzy strongly upper irresolute at $x \in X$.

**Theorem 3.8.**
For a fuzzy multifunction $F: X \to Y$ , the following are equivalent :
(a) $F$ is fuzzy strongly upper irresolute.
(b) For each fuzzy set $B$ of $Y$, $F^{+}(B) \in SO(X)$.
(c) For each point $x \in X$ and any fuzzy set $B$ of $Y$ with $F(x) \le B$, there exists $U \in SO(X)$ containing $x$ such that $F(U) \le B$.
(d) For each fuzzy set $B$ of $Y$, $F^{-}(B)$ is semiclosed in $X$.
(e) For each fuzzy set $B$ of $Y$, $int\left(cl\left(F^{-}(B)\right)\right) \subset F^{-}(B)$.
(f) For each fuzzy set $B$ of $Y$, $scl\left(F^{-}(B)\right) \subset F^{-}(B)$.
(g) For each fuzzy set $B$ of $Y$, $sint\left(F^{+}(B)\right) \supset F^{+}(B)$.

**Proof.**
(a) ⇒ (b) : Let $B$ be any fuzzy set of $Y$ with $x \in F^{+}(B)$. Then $F(x) \le B$. By Theorem 3.7 (a) ⇒ (b), $x \in cl\,(int(F^{+}(B)))$. Therefore, $F^{+}(B) \subseteq cl\,(int(F^{+}(B)))$. Hence $F^{+}(B) \in SO(X)$.

(b) ⇒ (c) : Let $x \in X$ and $B$ be any fuzzy set of $Y$ with $F(x) \le B$. Then $x \in F^{+}(B)$. Put $U = F^{+}(B)$. Then by (b), $U \in SO(X)$ containing $x$ and $F(U) = FF^{+}(B) = \{F(x): x \in F^{+}(B)\} = \{F(x): F(x) \le B\}$ and so $F(U) \le B$.

(c) ⇒ (d) : Let $B$ be any fuzzy set of $Y$. Let $x \in X\backslash F^{-}(B) = F^{+}(1_Y\backslash B)$. Then $F(x) \le (1_Y\backslash B)$. Then by (c), there exists $U \in SO(X)$ containing $x$ such that $F(U) \le (1_Y\backslash B)$. Therefore, $U \subseteq F^{+}(1_Y\backslash B) = X\backslash F^{-}(B)$ and hence $x \in U \subseteq cl(int\,U) \subseteq cl(int(X\backslash F^{-}(B)))$. Hence $int\left(X\backslash F^{-}(B)\right) \subseteq X\backslash F^{-}(B) \subseteq cl\,(int(X\backslash F^{-}(B)))$. Therefore, $X\backslash F^{-}(B)$ is semiopen in $X$ and hence $F^{-}(B)$ is semiclosed in $X$.

(d) ⇒ (e) : Let $B$ be any fuzzy set of $Y$. Then by (d), $F^{-}(B)$ is semiclosed in $X$ and hence $int\left(cl\left(F^{-}(B)\right)\right) \subset F^{-}(B)$.

(e) ⇒ (f) : Let $B$ be any fuzzy set of $Y$. We know that $scl\left(F^{-}(B)\right) = F^{-}(B) \cup int\left(cl\left(F^{-}(B)\right)\right) \subset F^{-}(B) \cup F^{-}(B)$ (by (e)) = $F^-(B)$.

(f) $\Rightarrow$ (g) : Let $B$ be any fuzzy set of $Y$. Then $X \setminus F^+(B) = F^-(1_Y \setminus B) \supset scl\, F^-(1_Y \setminus B)$ (by (f)) $= scl\left(X \setminus F^+(B)\right) = X \setminus sint\, F^+(B)$. Hence $sint\, F^+(B) \supset F^+(B)$.

(g) $\Rightarrow$ (a) : Let $x \in X$ and $B$ be any fuzzy set of $Y$ such that $F(x) \leq B$. Then $x \in F^+(B) \subseteq sint\, F^+(B)$ (by (g)) $= \cup\{U \in SO(X): U \subset F^+(B)\}$ and hence there exists $U \in SO(X)$ containing $x$ such that $F(U) \leq B$. Hence $F$ is fuzzy strongly upper irresolute.

**Theorem 3.9.**
If a fuzzy multifunction $F : X \longrightarrow Y$ is fuzzy strongly upper irresolute and fuzzy strongly lower semi-continuous, then it is fuzzy strongly irresolute.

**Proof.**
Let $B_1$, $B_2$ be any two fuzzy sets in $Y$ and $x \in F^+(B_1)$. Then $F(x) \leq B_1$ and since $F$ is fuzzy strongly upper irresolute, there exists $U \in SO(X)$ containing $x$ such that $F(U) \leq B_1$. Therefore, $x \in U \subseteq cl\,(intU) \subseteq cl\,(int\left(F^+(B_1)\right))$. Hence, $F^+(B_1) \subseteq cl\,(int\left(F^+(B_1)\right))$ and $F^+(B_1) \in SO(X)$. Since $F$ is fuzzy strongly lower semi-continuous, $F^-(B_2)$ is open in $X$. Therefore, $F^+(B_1) \cap F^-(B_2) \in SO(X)$. It then follows from Theorem 2.6 that $F$ is fuzzy strongly irresolute.

**Theorem 3.10.**
If a fuzzy multifunction $F : X \longrightarrow Y$ is fuzzy stongly lower irresolute and fuzzy strongly upper semi-continuous, then it is fuzzy strongly irresolute.

**Proof.**
Let $B_1$, $B_2$ be any two fuzzy sets in $Y$ and $x \in F^-(B_2)$. Then $F(x)q\,B_2$ and since $F$ is fuzzy strongly lower irresolute, there exsits $U \in SO(X)$ containing $x$ such that $F(u)q\,B_2$, for each $u \in U$. Then $u \in F^-(B_2)$, for all $u \in U \Rightarrow U \subset F^-(B_2)$. Therefore, $x \in U \subseteq cl\,(intU) \subseteq cl\,(int\left(F^-(B_2)\right))$. Hence, $F^-(B_2) \subseteq cl\,(int\left(F^-(B_2)\right))$ and so $F^-(B_2) \in SO(X)$. Since $F$ is fuzzy strongly upper semi-continuous, $F^+(B_1)$ is open in $X$ and hence $F^+(B_1) \cap F^-(B_2) \in SO(X)$. Then by Theorem 2.6 that $F$ is fuzzy strongly irresolute.

**Remark 3.11.**

(a) Every fuzzy strongly irresolute multifunction is fuzzy strongly upper irresolute as well as fuzzy strongly lower irresolute.

(b) If a fuzzy multifunction is fuzzy strongly upper as well as lower semi-continuous, then it is fuzzy strongly irresolute.

**Remark 3.12.**
$F$ is fuzzy strongly upper (lower) semi-continuous $\Rightarrow$ $F$ is fuzzy strongly upper (lower) irresolute. But the converse is not true, in general.

**Example 3.13.**
Let $= \{a, b, c\}$ , $\tau = \{\emptyset, X, \{a\}\}$, $Y = \{p\}$, $\tau_1 = \{0_Y, 1_Y\}$. Then $(X, \tau)$ be a topological space and $(Y, \tau_1)$ is a fuzzy topological space. Let $F : (X, \tau) \rightarrow (Y, \tau_1)$ be a fuzzy multifunction defined by $F(a) = A, F(b) = B$ and $F(c) = C$ where $A(p) = 0.45, B(p) = 0.4, C(p) = 0.61$. Let $D$ be any fuzzy set of $Y$.
If $0.55 < D(p) \leq 0.6$, then $F^-(D) = \{a, c\} \in SO(X)$, but $F^-(D) \notin \tau$. If $D(p) > 0.6, F^-(D) = X \in \tau$ and hence $F^-(D) \in SO(X)$. If $0.39 < D(p) \leq 0.55$, $F^-(D) = \{c\} \in \tau$ and hence $F^-(D) \in SO(X)$. If $0 \leq D(p) \leq 0.39$, $F^-(D) = \emptyset \in \tau$ and hence $F^-(D) \in SO(X)$. Therefore, $F$ is fuzzy strongly lower irresolute, but not fuzzy strongly lower semi-continuous.

**Example 3.14.**
Consider Example 3.13.. Let $D$ be any fuzzy set of $Y$. If $0.4 \leq D(p) < 0.61$, then $F^+(D) = \{a, b\} \in SO(X)$ but $F^+(D) \notin \tau$. If $D(p) \geq 0.61$, $F^+(D) = X \in \tau$ and hence $F^+(D) \in SO(X)$. If $0 \leq D(p) < 0.4$, $F^+(D) = \emptyset \in \tau$ and hence $F^+(D) \in SO(X)$. Therefore, $F$ is fuzzy strongly upper irresolute but not fuzzy strongly upper semi-continuous.

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