ISSN 2278-7763

CERTAIN SUFFICIENCY CONDITIONS PERTAINING TO GENERALIZED FOX-WRIGHT HYPERGEOMETRIC FUNCTIONS

V.B.L. CHAURASIA
Department of Mathematics, University of Rajasthan
Jaipur-302055 (Rajasthan), India

AND

R.C. MEGHWAL
Department of Mathematics, Government Post-Graduate College
Neemuch-458441 (MP) India

ABSTRACT

The aim of this paper is to establish some conditions for the functions $z\{{}_p\overline{\psi}_q(z)\}$ to be a member of certain subclasses of regular functions. The results derived here are of general nature and hence encompass several cases of interest hitherto scattered in the literature. [For example Swaminathan [19], Chaurasia and Srivastava [3]].

1. INTRODUCTION

The ${}_p\overline{\psi}_q$-generalized Fox-wright hypergeometric function appearing in the paper will be defined and represented in the following way [9]

$${}_p\overline{\psi}_q(z) = {}_p\overline{\psi}_q(z)\left[\begin{matrix}(e_j,E_j;A_j)_{1,p};\\(g_j,G_j;B_j)_{1,q};\end{matrix}\; z\right] \qquad \ldots(1)$$

it reduces to ${}_p\psi_q$, the Fox-Wright function ([22], p.50, equation (1.5)).

The series representation for ${}_p\overline{\psi}_q$ will be defined and represented as

$${}_p\overline{\psi}_q(z) = {}_p\overline{\psi}_q\left[\begin{matrix}(e_j,E_j;A_j)_{1,p};\\(g_j,G_j;B_j)_{1,q};\end{matrix}\; z\right]$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \{\Gamma(e_j + n E_j)\}^{A_j} z^n}{\prod_{j=1}^{q} \{\Gamma(g_j + n G_j)\}^{B_j}\, n!} \qquad \ldots(2)$$

Now, we describe an important case of ${}_p\overline{\Psi}_q$ that generalized several special functions

$${}_p\overline{F}_q\left[\begin{matrix}(e_j,1;A_j)_{1,p};\\(g_j,1;B_j)_{1,q};\end{matrix}\,z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\{\Gamma(e_j+n)\}^{A_j}\;z^n}{\prod_{j=1}^{q}\{\Gamma(g_j+n)\}^{B_j}\;n!} \qquad \ldots(3)$$

As usual, let A denote the class of functions of the form

$$f(z)=z+\sum_{\ell=2}^{\infty}a_\ell z^\ell \qquad \ldots(4)$$

regular in the open unit disk Δ = {z : | z | < 1 } and S denote the subclass of A that are univalent in Δ.

2. Now we start with the following definitions and lemmas.

DEFINITION 1 ([2]). Let f ∈ A, $0 \le \ell < \infty$, and $0 \le E < 1$. Then f ∈ ℓ - UCV(E) if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\ge \ell\left|\frac{zf''(z)}{f'(z)}\right|+E. \qquad \ldots(5)$$

This class generalizes various other classes which are suitable to mention. The class ℓ − UCV(0) called the ℓ−uniformly convex is due to Kanas and Wisniowska [10] and has its geometric characterization gives in the following way : let $0 \le \ell < \infty$. The function f ∈ A is said to be ℓ−uniformly convex in Δ, if f is convex in Δ and the image of every circular are V contained in Δ, with center ξ, where | ξ | ≤ ℓ, is convex. The class 0 − UCV(E) = ℓ(E) is well known class of convex functions of order E that satisfy the regular conditions

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}> E.$$

In particular, for E = 0, f maps the unit disk onto a convex domain (for details, see [7]). We have 1 − UCV(0) = UCV [8]. Denoting

$$p(z) = \left\{ 1 + \frac{z f^{''}(z)}{f^{'}(z)} \right\} (z \in \Delta)$$

and assuming that f ∈ UCV(E). We have that p is in conic region

$$\Omega = \{ w \in C : (\mathrm{Im}\, w)^2 < 2\,\mathrm{Re}\,(w) - 1 \}.$$

The classes UCV(E) and ST(E) are unified and studied using certain fractional calculus operator methods in [15]. We refer to [10,11,12] and references there in for basic results related to this paper.

For ∈ C\ {0}, Swaminathan [19] introduce the class $P_{\gamma}^{\Im}(G)$ with $0 \le \gamma < 1$ and G < 1, as

$$P_{\gamma}^{\Im}(G) = \left\{ f \in A : \left| \frac{(1-\gamma)\frac{f(z)}{z} + \gamma f^{'}(z) - 1}{2\Im\,(1-G) + (1-\gamma)\frac{f(z)}{z} + \gamma f^{'}(z) - 1} \right| < 1,\ z \in \Delta \right\}. \quad \ldots(6)$$

We list a few particular cases of this class discussed in the literature.

(i) The class $P_{\gamma}^{\Im}(G)$, with $0 \le \gamma < 1$ and G < 1 was studied by Dixit and Pal in [4]. Properties of that class related to the operator $I_{e,g}; c(f)(z) = z\,F(e,g;c;z) * f(z)$ were considered in [6].

(ii) The class $P_{\gamma}^{\Im}(G)$, with $0 \le \gamma < 1$ and G < 1 for $\Im = e^{i\eta}\cos\eta$ where $-\pi/2 < \eta < \pi/2$ was examined in [14] and discussed by several researcher with the references to the Carlson-Schaffer operator $G_g, c(f)(z) = z\,F(1,g;c;z) * f(z)$ using duality techniques (for example, see [1,5,13].

We denote by T a subclass S with negative coefficients, e.g.

$$f(z) = z - \sum_{n=2}^{\infty} a_{\ell} z^{\ell}, \quad a_{\ell} \ge 0 \quad \ldots(7)$$

This class is due to H.Silverman [20] has many interesting results ([20] and [21]).

In the link of ℓ − UCV(E) the following class was defined in [2].

DEFINITION 2. ([2]).

Let ℓ−UCT(E) be the class of functions f(z) of the form (8) that satisfy the condition (6) and an Alexander type theorem, the following classes are defined in [2].

DEFINITION 3 ([2]).

Let $0 \le \ell < \infty$, and $0 \le E < 1$. Then

(i) $f \in \ell - ST(E)$ if and only if f has the form (5) and satisfies the condition

$$\operatorname{Re}\left\{\frac{z f'(z)}{f(z)}\right\} \ge \ell \left|\frac{z f'(z)}{f(z)} - 1\right| + E \qquad \ldots(8)$$

(ii) $f \in \ell - STT(E)$ if and only if f has the form (7) and satisfies the inequality given by the expression (8). For $\ell = 0$, we get the well known class of starlike functions of order E, which has the regular characterization

$$\operatorname{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > E \ (z \in \Delta)$$

In particular, for E = 0, f maps the unit disk onto a starlike domain (for details, see [5]. We further note that, 1 − ST(E) is the well known class discussed in [18]. We also need the following sufficient condition on the coefficients for the functions in the class ℓ − UCV(E).

LEMMA 1 ([2]). If f is of the form (5) satisfies a condition

$$\sum_{n=2}^{\infty} n\,[n(\ell+1) - (\ell+E)]|e_n| \le 1 - E \qquad \ldots(9)$$

for some $0 \le \ell < \infty$ and $E \in [0,1)$, then $f \in \ell$ − UCV(E). Moreover, the above condition is necessary and sufficient for f to be in ℓ − UCT(E), further the condition

$$\sum_{n=2}^{\infty} [n(\ell+1) - (\ell+E)]\,|e_n| \le 1 - E \qquad \ldots(10)$$

is sufficient for f to be in ℓ−ST(E) and it is both necessary and sufficient for f to be in ℓ−STT(E). Another sufficient condition is also given for the class ℓ−UCV(E) in [10] which is given as follows

LEMMA 2 ([10]). If $f \in S$ and be of the form (4) satisfies a condition

$$\sum_{n=2}^{\infty} n(n-1)|e_n| \le \frac{1}{\ell+2} \quad \ldots(11)$$

for some ℓ , $0 \le \ell < \infty$ then $f \in \ell - UCV(E)$. The number $\frac{1}{\ell+2}$ can not be increased.

LEMMA 3 ([19). If f of the form (4) satisfies a sufficient condition

$$\sum_{n=2}^{\infty} [1+\gamma(n-1)]\,|e_n| \le |\Im|\,(1-G) \quad \ldots(12)$$

then $f \in P_{\gamma}^{\Im}(G)$. This condition is also necessary if f is of the form (7) and $\Im = 1$.

3. MAIN RESULTS

THEOREM (3.1). If $A_j > 0, B_j > 0, \sum_{j=1}^{q} g_j > \sum_{j=1}^{p} e_j + 2, e_j > 0$ and $1+\sum_{j=1}^{q} B_j G_j > \sum_{j=1}^{p} A_j E_j$, then a sufficient condition for the function $z\{{}_p\overline{\Psi}_q(z)\}$ to be in the class ℓ–UCV(E), and both necessary and sufficient conditions for $z\{{}_p\overline{\Psi}_q(z)\}$ to be in ℓ–UCT(E), $(0 \le \ell < \infty; 0 \le E < 1)$ is

$$\left(\frac{1+\ell}{1-E}\right) {}_p\overline{\Psi}_q\left[\begin{matrix}(e_j+2E_j,E_j;A_j)_{1,p};\\(g_j+2G_j,G_j;B_j)_{1,q};\end{matrix} 1\right] + \left(\frac{3+2\ell-E}{1-E}\right) {}_p\overline{\Psi}_q\left[\begin{matrix}(e_j+E_j,E_j;A_j)_{1,p};\\(g_j+G_j,G_j;B_j)_{1,q};\end{matrix} 1\right]$$

$$+ {}_p\overline{\Psi}_q\left[\begin{matrix}(e_j,E_j;A_j)_{1,p};\\(g_j,G_j;B_j)_{1,q};\end{matrix} 1\right] \le 1 + \frac{\prod_{j=1}^{p} \{\Gamma(e_j)\}^{A_j}}{\prod_{j=1}^{q} \{\Gamma(g_j)\}^{B_j}} \quad \ldots(13)$$

Proof. By using LEMMA 1, equation (9), it is sufficient to prove that

$$\sum_{n=2}^{\infty} n\,[n(1+\ell)-(\ell-E)]\left[\frac{\prod_{j=1}^{p} \{\Gamma[e_j+E_j(n-1)]\}^{A_j}}{\prod_{j=1}^{q} \{\Gamma[g_j+G_j(n-1)]\}^{B_j}} \frac{1}{n-1!}\right] \le (1-E). \quad \ldots(14)$$

The above inequality may be expressed as

$$\sum_{n=1}^{\infty} (n+1)^2 (1+\ell) \left[\frac{\prod_{j=1}^{p} \{\Gamma[e_j + E_j n]\}^{A_j}}{\prod_{j=1}^{q} \{\Gamma[g_j + G_j n]\}^{B_j} n!} \right]$$

$$-\sum_{n=1}^{\infty} (n\ell + nE + \ell + E) \frac{\prod_{j=1}^{p} \{\Gamma[e_j + E_j n]\}^{A_j}}{\prod_{j=1}^{q} \{\Gamma[g_j + G_j n]\}^{B_j} n!} \le (1-E).$$

Now the left hand side of the above inequality is equal to

$$(1+\ell)\sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p} \{\Gamma[e_j + E_j n]\}^{A_j}}{\prod_{j=1}^{q} \{\Gamma[g_j + G_j n]\}^{B_j}} \frac{1}{(n-2)!}$$

$$+(3+2\ell-E)\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \{\Gamma[e_j + (n+1)E_j]\}^{A_j}}{\prod_{j=1}^{q} \{\Gamma[g_j + (n+1)G_j]\}^{B_j}} \frac{1}{n!}$$

$$+(1-E)\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{p} \{\Gamma[e_j + nE_j]\}^{A_j}}{\prod_{j=1}^{q} \{\Gamma[g_j + nG_j]\}^{B_j}} \frac{1}{n!} \le (1-E)$$

ISSN 2278-7763

$$(1+\ell)\,{}_p\overline{\Psi}_q\left[\begin{matrix}(e_j+2E_j,E_j;A_j)_{1,p};\\(g_j+2G_j,G_j;B_j)_{1,q};\end{matrix}1\right]$$

$$+(3+2\ell-E)\,{}_p\overline{\Psi}_q\left[\begin{matrix}(e_j+E_j,E_j;A_j)_{1,p};\\(g_j+G_j,G_j;B_j)_{1,q};\end{matrix}1\right]$$

$$+(1-E)\,{}_p\overline{\Psi}_q\left[\begin{matrix}(e_j,E_j;A_j)_{1,p};\\(g_j,G_j;B_j)_{1,q};\end{matrix}1\right]-\frac{\prod_{j=1}^{p}\{\Gamma(e_j)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(g_j)\}^{B_j}}\le(1-E)$$

and is bounded by (1 – E) iff (13) holds. It ends the theorem.

THEOREM 3.2. If $A_j>0, B_j>0, \sum_{j=1}^{q} g_j>\sum_{j=1}^{p} e_j+1, e_j>0$ and $1+\sum_{j=1}^{q} B_jG_j>\sum_{j=1}^{p} A_jE_j$, then a sufficient condition for the function $z\{{}_p\overline{\Psi}_q(z)\}$ to be in the class $\ell-ST(E)$ and it is both necessary and sufficient condition for $z\{{}_p\overline{\Psi}_q(z)\}$ to be in $\ell-STT(E)$ $(0\le\ell<\infty, 0\le E<1)$ is

$$\left(\frac{1+\ell}{1-E}\right){}_p\overline{\Psi}_q\left[\begin{matrix}(e_j+E_j,E_j;A_j)_{1,p};\\(g_j+G_j,G_j;B_j)_{1,q};\end{matrix}1\right]$$

$$+{}_p\overline{\Psi}_q\left[\begin{matrix}(e_j,E_j;A_j)_{1,p};\\(g_j,G_j;B_j)_{1,q};\end{matrix}1\right]\le 1+\frac{\prod_{j=1}^{p}\{\Gamma(e_j)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(g_j)\}^{B_j}}. \qquad \ldots(15)$$

Proof. Since

$$z\{{}_p\overline{\Psi}_q(z)\}=\frac{\prod_{j=1}^{p}\{\Gamma(e_j)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(g_j)\}^{B_j}}z$$

$$+\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{p}\{\Gamma[e_j+(n-1)E_j]\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma[g_j+(n-1)G_j]\}^{B_j}}\frac{z^n}{(n-1)!}$$

Now by using Lemma 1 (equation (10)), we only require to show that

$$\sum_{n=2}^{\infty}[n(\ell+1)-(\ell+E)]\left[\frac{\prod_{j=1}^{p}\{\Gamma[e_j+(n-1)E_j]\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma[g_j+(n-1)G_j]\}^{B_j}}\frac{1}{(n-1)!}\right]\le(1-E) \qquad \ldots(16)$$

Now, we have

$$\sum_{n=0}^{\infty}[(n+2)(\ell+1)-(\ell+E)]\left[\frac{\prod_{j=1}^{p}\{\Gamma[e_j+(n+1)E_j]\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma[g_j+(n+1)G_j]\}^{B_j}(n+1)!}\right]\le(1-E)$$

$$=(\ell+1)\left[\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\{\Gamma[e_j+E_j)+nE_j]\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma[g_j+G_j)+nG_j]\}^{B_j}!}\frac{1^n}{n!}\right]$$

$$+(1-E)\left[\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\{\Gamma[e_j+nE_j]\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma[g_j+nG_j]\}^{B_j}!}\frac{1^n}{n!}-\frac{\prod_{j=1}^{p}\{\Gamma(e_j)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(g_j)\}^{B_j}}\right]\le(1-E)$$

$$=(\ell+1)\,{}_p\overline{\Psi}_q\left[\begin{matrix}(e_j+E_j,E_j;A_j)_{1,p};\\(g_j+G_j,G_j;B_j)_{1,q};\end{matrix}\,1\right]+(1-E)\,{}_p\overline{\Psi}_q\left[\begin{matrix}(e_j+E_j;A_j)_{1,p};\\(g_j+G_j;B_j)_{1,q};\end{matrix}\,1\right]$$

$$-(1-E)\frac{\prod_{j=1}^{p}\{\Gamma(e_j)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(g_j)\}^{B_j}}\le(1-E)$$

ISSN 2278-7763

by the assertion (15). Hence $z\{{}_p\overline{\Psi}_q(z)\} \in \ell - ST(E)$.

Remark 1. When $A_j = B_j = 1$ and $\ell = 2 - E$, Theorem 3.2 corresponds to a result given earlier by Chaurasia and Srivastav [[3], p.2, Theorem 2.1]

THEOREM (3.3). If $A_j > 0, B_j > 0, \sum_{j=1}^{q} g_j > \sum_{j=1}^{p} e_j + 1, e_j > 0$ and $1 + \sum_{j=1}^{q} B_j G_j > \sum_{j=1}^{p} A_j E_j$, then a sufficient condition for the function $z\{{}_p\overline{\Psi}_q(z)\}$ to be in the class ℓ–UCV(E) $(0 \le \ell < \infty)$ is

$$(\ell+2)\,{}_p\overline{\Psi}_q\left[\begin{matrix}(e_j+2E_j,E_j;A_j)_{1,p};\\(g_j+2G_j,G_j;B_j)_{1,q};\end{matrix}\,1\right]$$

$$+2(\ell+2)\,{}_p\overline{\Psi}_q\left[\begin{matrix}(e_j+E_j\},E_j;A_j)_{1,p};\\(g_j+G_j|,G_j;B_j)_{1,q};\end{matrix}\,1\right] \le 1 \qquad \ldots(17)$$

The number $\frac{1}{\ell+2}$ can not be increased.

Proof. By using Lemma 2, equation (11), it is sufficient to prove that,

$$\sum_{n=2}^{\infty}\left|\frac{\prod_{j=1}^{p}\{\Gamma[e_j+(n-1)E_j\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma[g_j+(n-1)G_j\}^{B_j}}\,\frac{1}{(n-1)!}\right| \le \frac{1}{\ell+2}\,. \qquad \ldots(18)$$

We note that

$$\sum_{n=1}^{\infty}\{(n+1)^2-(n+1)\}\left|\frac{\prod_{j=1}^{p}\{\Gamma[e_j+nE_j]\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma[g_j+nG_j]\}^{B_j}}\,\frac{1}{n!}\right| \le \frac{1}{\ell+2}$$

$$=\sum_{n=2}^{\infty}\left|\frac{\prod_{j=1}^{p}\{\Gamma[e_j+nE_j]\}^{A_j}\,1}{\prod_{j=1}^{q}\{\Gamma[g_j+nG_j]\}^{B_j}\,(n-2)!}\right|$$

$$+2\sum_{n=1}^{\infty}\left|\frac{\prod_{j=1}^{p}\{\Gamma[e_j+nE_j]\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma[g_j+nG_j]\}^{B_j}}\frac{1}{(n-1)!}\right|\leq\frac{1}{\ell+2}$$

$${}_p\overline{\psi}_q\left[\begin{matrix}(e_j+2E_j|,E_j;A_j)_{1,p};\\(g_j+2G_j|,G_j;B_j)_{1,q};\end{matrix}1\right]+2\,{}_p\overline{\psi}_q\left[\begin{matrix}(E_j,A_j)_{1,p};\\(G_j;B_j)_{1,q};\end{matrix}1\right]\leq\frac{1}{\ell+2}$$

The left hand side is bounded by $\frac{1}{\ell+2}$ if (17) holds. Hence the theorem is proved.

REMARK 2. When

$$A_j=B_j=1, p=2, q=1, E_1=E_2=G_1=1, e_1=e, e_2=g, g_1=c,\; E=0 \qquad \ldots(19)$$

and other some manipulation, Theorem (3.3) provides a similar results obtained earlier by Swaminathan [[19], p.6, Theorem 2.10, Corollary 2.11, Corollary 2.12].

THEOREM 3.4. If $A_j>0, B_j>0, \sum_{j=1}^{q} g_j>\sum_{j=1}^{p} e_j+1, e_j>0$ and $1+\sum_{j=1}^{q} B_jG_j>\sum_{j=1}^{p} A_jE_j$, then a sufficient condition for the function $z\{{}_p\overline{\psi}_q(z)\}$ to be in the class $P_\gamma(G)$ is

$$\gamma\,{}_p\overline{\psi}_q\left[\begin{matrix}(e_j+E_j|,E_j;A_j)_{1,p};\\(g_j+G_j|,G_j;B_j)_{1,q};\end{matrix}1\right]$$

$$+{}_p\overline{\psi}_q\left[\begin{matrix}(e_j|,E_j;A_j)_{1,p};\\(g_j|,G_j;B_j)_{1,q};\end{matrix}1\right]\leq|\Im|(1-G)+\frac{\prod_{j=1}^{p}\{\Gamma(e_j)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(g_j)\}^{B_j}} \qquad \ldots(20)$$

Proof. By using Lemma (3) (equation (12)) it is sufficient to prove that

$$\sum_{n=2}^{\infty}[1+\gamma(n-1)]\left|\frac{\prod_{j=1}^{p}\{\Gamma[e_j+(n-1)E_j]\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma[g_j+(n-1)G_j\}^{B_j}}\frac{1}{(n-1)!}\right| \leq (1-G) \qquad \ldots(21)$$

Now, we have

$$\sum_{n=2}^{\infty}[n\gamma+(1-\gamma)]\left|\frac{\prod_{j=1}^{p}\{\Gamma[e_j+(n-1)E_j]\}^{A_j}1}{\prod_{j=1}^{q}\{\Gamma[g_j+(n-1)G_j\}^{B_j}(n-1)!}\right|$$

$$=\sum_{n=0}^{\infty}(n+1)\gamma\left|\frac{\prod_{j=1}^{p}\{\Gamma[e_j+(n+1)E_j]\}^{A_j}1}{\prod_{j=1}^{q}\{\Gamma[g_j+(n+1)G_j\}^{B_j}(n+1)!}\right|$$

$$+\sum_{n=0}^{\infty}\left|\frac{\prod_{j=1}^{p}\{\Gamma[e_j+nE_j]\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma[g_j+nG_j\}^{B_j}}\frac{1}{n!}\right|-\frac{\prod_{j=1}^{p}\{\Gamma(e_j)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(g_j)\}^{B_j}}$$

which is bounded by | | (1 – G)if (20) holds. It ends the proof.

REMARK 3. When A_j = B_j 1 and applying the parameter substitution listed in [20] and after some manipulation, Theorem (3.4) would yield the similar known results due to Swaminathan [[19], p.5, Theorem 2.5, Corollary 2.6, Theorem 2.7, Theorem 2.8, Corollary 2.9].

4. PARTICULAR CASES

When A_j = B_j = 1 and specifying the parameters appropriately, the results of this paper immediately yields some results due to Swaminathan [19], Chaurasia and Srivastav [3].

5. SPECIAL CASES

A. THEOREM (5.1). By taking $E_j = G_j = 1$ in the Theorem (3.1) $A_j > 0$, $B_j > 0$, $\sum_{j=1}^{q} g_j > \sum_{j=1}^{p} e_j + 2, e_j > 0$ and $1+q > p$ then a sufficient condition for the function $z\{{}_p\overline{F}_q(z)\}$ to be in the $\ell-UCV(E)$, and both necessary and sufficient conditions for $z\{{}_p\overline{F}_q(z)\}$ to be in $\ell-UCT(E)$, $(0 \le \ell < \infty, 0 \le E < 1)$ is

$$\left(\frac{1+\ell}{1-E}\right){}_p\overline{F}_q\left[\begin{matrix}(e_j+2;1;A_j)_{1,p};\\(g_j+2;1;B_j)_{1,q};\end{matrix}1\right]+\left(\frac{3+2\ell-E}{1-E}\right){}_p\overline{F}_q\left[\begin{matrix}(e_j+1,1;A_j)_{1,p};\\(g_j+1,1;B_j)_{1,q};\end{matrix}1\right]$$

$$+{}_p\overline{F}_q\left[\begin{matrix}(e_j,1;A_j)_{1,p};\\(g_j,1;B_j)_{1,q};\end{matrix}1\right]\le 1+\frac{\prod_{j=1}^{p}\{\Gamma(e_j)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(g_j)\}^{B_j}} \qquad \ldots(22)$$

(B) THEOREM (5.2). By taking $E_j = G_j = 1$ in the Theorem (3.2) $A_j > 0$, $B_j > 0$, $\sum_{j=1}^{q} g_j > \sum_{j=1}^{p} e_j + 1, e_j > 0$ and $1 + \sum_{j=1}^{q} B_j > \sum_{j=1}^{p} A_j$ then the sufficient condition for the function $z\{{}_p\overline{F}_q(z)\}$ to be in the $\ell-ST(E)$, and it is both necessary and sufficient conditions for $z\{{}_p\overline{F}_q(z)\}$ to be in $\ell-STT(E)$, $(0 \le \ell < \infty, 0 \le E < 1)$ is

$$\left(\frac{1+\ell}{1-E}\right){}_p\overline{F}_q\left[\begin{matrix}(e_j+1;1;A_j)_{1,p};\\(g_j+1;1;B_j)_{1,q};\end{matrix}1\right]+{}_p\overline{F}_q\left[\begin{matrix}(e_j,1;A_j)_{1,p};\\(g_j,1;B_j)_{1,q};\end{matrix}1\right]\le 1+\frac{\prod_{j=1}^{p}\{\Gamma(e_j)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(g_j)\}^{B_j}} \qquad \ldots(23)$$

(C) THEOREM (5.3). By taking $E_j = G_j = 1$ in the Theorem (3.3) $A_j > 0$, $B_j > 0$, $\sum_{j=1}^{q} g_j > \sum_{j=1}^{p} e_j + 1, e_j > 0$ and $1 + \sum_{j=1}^{q} B_j > \sum_{j=1}^{p} A_j$ then the sufficient condition for the function $z\{{}_p\overline{F}_q(z)\}$ to be in the class $\ell-UCV(E)$, $(0 \le \ell < \infty)$ is

$$(\ell+2)\ {}_p\overline{F}_q\left[\begin{matrix}(e_j+2;1;A_j)_{1,p};\\(g_j+2;1;B_j)_{1,q};\end{matrix}1\right]+2(\ell+2)\ {}_p\overline{F}_q\left[\begin{matrix}(e_j+1,1;A_j)_{1,p};\\(g_j+1,1;B_j)_{1,q};\end{matrix}1\right]\le 1 \qquad \ldots(24)$$

the number $\frac{1}{\ell+2}$ can not be increased.

(D) THEOREM (5.4). By taking $E_j = G_j = 1$ in the Theorem (3.4) $A_j > 0$, $B_j > 0$, $\sum_{j=1}^{q} g_j > \sum_{j=1}^{p} e_j + 1$, $e_j > 0$ and $1 + \sum_{j=1}^{q} B_j > \sum_{j=1}^{p} A_j$ then the sufficient condition for the function $z\{{}_p\overline{F}_q(z)\}$ to be in the class $P_\gamma^{\Im}(G)$ is

$$\gamma\ {}_p\overline{F}_q\left[\begin{matrix}(e_j+1;1;A_j)_{1,p};\\(g_j+1;1;B_j)_{1,q};\end{matrix}1\right]+{}_p\overline{F}_q\left[\begin{matrix}(e_j,1;A_j)_{1,p};\\(g_j,1;B_j)_{1,q};\end{matrix}1\right]\le |\Im|(1-G)+\frac{\prod_{j=1}^{p}\{\Gamma(e_j)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(g_j)\}^{B_j}} \qquad \ldots(25)$$

which is bounded by | | (1 – G) if (25) holds.

NOTE: The function ${}_p\overline{F}_q$ reduces to well known ${}_pF_q$ for $A_j = 1$ (j = 1,…,p), $B_j = 1$(j = 1,…,q) in it.

ACKNOWLEDGEMENT

The authors are thankful to Professor H.M. Srivastava, University of Victoria, Canada for his kind help and valuable suggestions in the preparation of this paper.

REFERENCES

[1] Balasubramanian, R., Ponnusamy, S. and Vuorinen, M.V., On hypergeometric functions and function spaces, J. Comput. Appl. Math. 139 (2002), 299-322.

[2] Bharati, R., Parvatham, R. and Swaminathan, A., On subclasses of uniformly convex functions and corresponding class of starlike functions, Tamkang J. Math. 28 (1997), 17-32.

[3] Chaurasia, V.B.L. and Srivastava, A., Uniformly starlike and uniformly convex functions pertaining to special functions, J. Ineq. Pure Appl. Math. 9(1), 30 (2008).

[4] Dixit, K.K. and Pal, S.K., On a class of univalent functions related to complex order, Indian J. Pure Appl. Math. 26(9) 1995, 889-896.

[5] Fourier, R. and Ruschweyh, St., On two extremal problems related to univalent functions, Rocky Mountain J. Math. 24 (1994), 529-538.

[6] Gangadharan, A., Shanmugam,T.N. and Srivastava, H.M., Generalized hypergeometric function associated functions associated with K-uniformly convex function, Comput. Math. Appl. 44 (2002), 1515-1526.

[7] Goodman, A.W., Univalent functions Vols. I and II, Polygonal Publishing House, Washington, New Jersey, 1983.

[8] Goodman, A.W., On uniformly convex functions, Ann. Polon. Math. 56 (1991), 87-92.

[9] Gupta, K.C., Jain, Rashmi and Sharma, Aarti, A study of unified finite integral transform with application, J. Rajasthan Acad. Phy. Sci., Vol.2, No.4, Dec. 2003, pp.269-282.

[10] Kanas, S. and Wisniwoska, A., Conic regions and K-uniform convexity, J. Comput. Appl. Math. 105(1999), 327-336.

[11] Kanas, S. and Klisniwaska, A., Conic region and K-starlike functions, Rev. Roumaine Math. Pures Appl. 45(2000), 647-657.

[12] Kanas, S. and Srivastava, H.M., Linear operators associated with K-uniformly convex functions, integral transform. Spec. funct. 9(2000), 121-132.

[13] Kim, Y.C. and Srivastava, H.M., Fractional integral and other linear operators associated with the Gaussian hypergeometric functions, Complex variable theory Appl. 34(1997), 293-312.

[14] Kim, Y.C. and Ronning, F., Integral transform of certain subclasses of analytic functions, J. Math. Anal. Appl. 258(2001), 466-486.

[15] Mishra, A.K. and Srivastava, H.M., Applications of fractional calculus to parabolic starlike and uniformly convex functions, Comput. Math. Appl. 39(3/4)(2000), 57-69.

[16] Ponnusamy, S. and Ronning, F., Duality for Hadmards products applied to certain integral transforms, Complex variable theory, Appl.32 (1997), 263-287.

[17] Ponnusamy, S., Hypergeometric transforms of functions with derivative in a half plane, J. Comput. Appl. Math. 96(1998), 35-49.

[18] Ronning, F., Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc. 18(1993), 189-196.

[19] Swaminathan, A., Certain sufficiency conditions on Gaussian hypergeometric functions, J. Ineq. Pure Appl. Math. 5(4), 83(2004).

[20] Silverman, H., Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51(1975), 109-116.

[21] Silverman, H., Convolutions of univalent function with negative coefficients, Ann. Univ. Mariae Curie-Skodowska Sect. A29(1975), 99-107.

[22] Srivastava, H.M. and Manocha, H.L., A treatise on generating functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

