Asymptotic Attractivity Results For Functional Differential Equation In Banach Algebras

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ABSTRACT

In this paper, we have proved an existence results for local asymptotic attractivity and existence of asymptotic stability of solutions for nonlinear functional differential equations in Banach algebras.

Keywords: Asymptotic attractivity and stability,Nonlinear functional differential equations ,Banach algebras.

1. Statement of Problem

LET $\mathbb{R}$ denote the real line and let $I_0 = [-r, 0]$ and $I = [0, a]$ be two closed and bounded intervals in $\mathbb{R}$. Let $J = I_0 \cup I$, then J is a closed and bounded intervals in $\mathbb{R}$. Let C denote the Banach space of all continuous real-valued functions ϕ on I_0 with the supremum norm $\|\cdot\|_c$ defined by

$$\|\phi\|_c = \sup_{t \in I_0} |\phi(t)|$$

Clearly C is a Banach algebras with respect to this norm and the multiplication" • " defined by

$$(x \cdot y)(t) = x(t) \cdot y(t), \quad t \in I_0$$

Consider the first order functional differential equation (In short FDE)

1.1 $$\frac{d}{dt}\left(\frac{x(t)}{f(t, x(t))}\right) = g(t, x_t) \quad \text{a.e. } t \in I$$

$$x(t) = \phi(t), \quad t \in I_0$$

Where $f: I \times \mathbb{R} \to \mathbb{R} - \{0\}$ is continuous, $g: I \times C \to \mathbb{R}$ and function $x_t(\theta) = x(t + \theta)$ for all $\theta \in I_0$

By a solution of FDE (1.1) we means,a function $x \in C(J, \mathbb{R}) \cap AC(I, \mathbb{R}) \cap C(I_0, \mathbb{R})$ that satisfies the equations in (1.1), where $AC(I, \mathbb{R})$ is the space of all absolutely continuous real-valued functions on J.

The functional differential equations have been the most active area of research since long time. See Hale (1), Henderson (2) and the references there in. But the study of functional differential equations in Banach algebras is very rare in the literature. Very resently the study along this line has been initiated via fixed point theorems. See Dhage and Regan (4) and Dhage(3) and the references there in. The FDE (1.1) is new to the literature and the study of this problem will definitely contribute immensely to the area of functional differential equations. In this paper, we proved the Local asymptotic attractivity results via a hybrid fixed point theorem of Dhage(5) which in turn gives the asymptotic stability of the solutions for the FDE(1.1). The results obtained in the paper generalize and extend several ones obtained earlier in a lot of paper concerning the asymptotic attractivity of solutions for some functional differential and integral equation.

2. Preliminaries and characterizations of solutions

Let X=$BC(J, \mathbb{R})$ be the space of continuous and bounded real-valued functions on J. Let Ω be a subset of X. Let$Q: X \to X$ be an operator and consider the following operator equation inX,

2.1 $$x(t) = Qx(t) \quad \text{for } t \in J.$$

2.2 Definition: We say that solutions of equation (2.1) are locally attractively if there exists a closed ball $\overline{\mathcal{B}}_r(x_0)$ in the space$BC(J, \mathbb{R})$, for some $x_0 \in BC(J, \mathbb{R})$ and $r > 0$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (2.1) belonging to $\overline{\mathcal{B}}_r(x_0) \cap \Omega$ we have

2.3 $lim_{t\to\infty}[x(t)-y(t)]=0$

In the case when this limit (2.3) is uniform with respect to the set $B(x_0,r)\cap\Omega$ i.e. when for each $\epsilon>0$, $\exists T>0$ such that

2.4 $|x(t)-y(t)|\le\epsilon$

For all $x,y\in B(x_0,r)\cap\Omega$ being solution of (2.1) and $t\ge T$, we will say that solutions of equation (2.1) are uniformly local attractive on J.

Here remark that the point x_0 belonging to the Banach space$BC(J,\mathbb{R})$ in the above definition is specific in the sense that (2.3) or (2.4) may not hold for any other point in $BC(J,\mathbb{R})$ different from x_0.

2.5 Definition: A solution$x\in BC(J,\mathbb{R})$ of equation (2.1) is said to be asymptotic if there exists a real number α such that $\lim_{t\to\infty}x(t)=\alpha$ and we say that the solution x is asymptotic to the number α on J.

We need the following definition used in the sequel.

2.6 Definition: The solutions of the equation (2.1) are said to be locally asymptotically attractive if there exists $x_0\in BC(J,\mathbb{R})$ and $r>0$ such that for all asymptotic solutions$x=x(t), y=y(t)$ of the equation (2.1) belonging to $B(x_0,r)\cap\Omega$, we have that condition (2.3) is satisfied. In the case when the condition (2.3) is satisfied uniformly with respect to the set $B(x_0,r)\cap\Omega$, we will say that solutions of the equation (2.1) are uniformly locally asymptotically attractive.

2.7 Remark: By $L^1(J,\mathbb{R})$ we denote the space of Lebesgue integrable functions on J and the norm $\|\cdot\|_{L^1}$ in $L^1(J,\mathbb{R})$ is defined by

$$\|x\|_{L^1}=\int_0^\infty|x(t)|\,ds$$

We employ a hybrid fixed point theorem of Dhage (5) for proving the existence results for uniform local asymptotic attracivity of the solutions for the FDE (1.1).

We will need following useful definition.

2.8 Definition: Let X be a Banach space with norm$\|\cdot\|$. An operator $Q:X\to X$ is called Lipschitz if there exists a constant $K>0$ such that

$$\|Qx-Qy\|\le K\|x-y\| \quad \text{for all} x,y\in X$$

The constant K is called the Lipschitz constant of Q on X.

2.9 Definition: An operator $Q:X\to X$ is called compact if $\overline{Q(S)}$ is a compact for any subset S of X. Similarly $Q:X\to X$ is called totally bounded if Q maps a bounded subset of X. Finally $Q:X\to X$ is called completely continuous operator if it is continuous and totally bounded operator on X.

2.10 Theorem [Dhage 5]: *Let S be a closed convex and bounded subset of the Banach algebra X and let $A,B:S\to X$ be two operators such that*

(a) *A is Lipschitz with Lipschit's constant K*
(b) *B is completely continuous*
(c) *$AxBx\in S$ for all $x\in S$, and*
(d) *$MK<1$, where $M=\|B(S)\|=sup\{\|Bx\|:x\in J\}$*
Then the operator equation

2.11 $AxBx=x$

has a solution and the set of all solutions is compact in S.

In the following section, we describe our hypotheses and prove the main results of this paper.

3. Existence of Solution and Local Asymptotic Attractivity

We consider the following of hypotheses in the sequel.

(H_1) The function $f:I\times\mathbb{R}\to\mathbb{R}-\{0\}$ is continuous and there exists a bounded function $\ell:\mathbb{R}_+\to\mathbb{R}_+$ with bound L satisfying

$|f(t,x)-f(t,y)|\le\ell(t)|x-y|$ for all $t\in I$ and $x,y\in\mathbb{R}$.

(H_2) The function $F:I\to\mathbb{R}$ defined by $F(t)=|f(t,o)|$ is bounded with $F_0=\sup_{t\ge0}F(t)$.

(H_3) The function $\beta:\mathbb{R}_+\to\mathbb{R}_+$ is continuous.

(H_4)The function $q:\mathbb{R}_+\to\mathbb{R}_+$ is continuous and $\lim_{t\to\infty}\phi(t)=0$

(H_5)The function $g:I\times C\to\mathbb{R}$ is continuous and there exists continuous function $a,b:\mathbb{R}_+\to\mathbb{R}_+$ satisfying

$|g(t,x)|\le a(t)b(s)$, for all $t,s\in\mathbb{R}_+=I$ and $x\in C$,

Where $\lim_{t\to\infty}a(t)\int_0^{\beta(t)}b(s)ds=0$

3.2 Theorem: *Assume that the hypotheses(H_1), (H_2) through $((H_5)$ hold. Furthermore, if $L(K_1+K_2)<1$, then the FDE [1.1] has at least one solution in the space $BC(J,\mathbb{R})$ and are solutions are uniformly locally* *asymptotically attractive on J, where constants $K_1 > 0, K_2 > 0$ are $K_1 = \lim_{t\to\infty} q(t)$ and $K_2 = \sup_{t\geq 0} \vartheta(t) = \sup_{t\geq 0}\left[a(t)\int_0^{\beta(t)} b(s)ds\right]$.*

Proof: Set X= $BC(J,\mathbb{R})$. Consider the closed ball $\overline{\mathcal{B}}_r(0)$ in X centered at origin O and radius r,

where $r = \frac{F_0(K_1+K_2)}{1-L(K_1+K_2)} > 0$

Now the FDE (1.1) is equivalent to the functional integral equation (In short FIE)

3.3 $x(t) = [f(t,x(t))]\left(\phi(0) + \int_0^{\beta(t)} g(s,x_s)ds\right)$, $t \in I$

and

3.4 $x(t) = \phi(t)$, $t \in I_0$

Define the two mapping A and B on $\overline{\mathcal{B}}_r(0)$ by

3.5 $Ax(t) = \begin{cases} f(t,x(t)) & \text{if } t \in I \\ 1 & \text{if } t \in I_0 \end{cases}$ and

3.6 $Bx(t) = \begin{cases} \phi(0) + \int_0^{\beta(t)} g(s,x_s)ds & \text{if } t \in I \\ \phi(t) & \text{if } t \in I_0 \end{cases}$

Obviously A and B define the operators $A, B: \overline{\mathcal{B}}_r(0) \to X$. We shall show that A and B satisfy all the conditions of Theorem (3.2) on$\overline{\mathcal{B}}_r(0)$.

We first show that A is a Lipschitzian on$\overline{\mathcal{B}}_r(0)$. Let $x, y \in \overline{\mathcal{B}}_r(0)$. Then by (H_1),

$$|Ax(t) - Ay(t)| \leq |f(t,x(t)) - f(t,y(t))|$$

$$\leq K(t)|x(t) - y(t)|$$

$$\leq K(t)\|x - y\| \quad \text{for all } t \in J.$$

Taking the supremum over t, we obtain

$$\|Ax - Ay\| \leq \|K\|\|x - y\| \quad \text{for all } x, y \in \overline{\mathcal{B}}_r(0)$$

So A is a Lipschitzian on $\overline{\mathcal{B}}_r(0)$ with a Lipschitz constant$\|K\|$.

Next we show that B is continuous and compact operator on $\overline{\mathcal{B}}_r(0)$. First we show that B is continuous on $\overline{\mathcal{B}}_r(0)$, for this $\in > 0$ and $x, y \in \overline{\mathcal{B}}_r(0)$ such that $\|x - y\leq\epsilon$. Then we get

$$|Bx(t) - By(t)| \leq \left|\phi(0) + \int_0^{\beta(t)} g(s,x_s)ds - \phi(0) - \int_0^{\beta(t)} g(s,y_s)ds\right|$$

$$\leq \int_0^{\beta(t)} |g(s,x_s) - g(s,y_s)|ds$$

$$\leq \int_0^{\beta(t)} \{|g(s,x_s)| + |g(s,y_s)|\}ds$$

$$\leq \int_0^{\beta(t)} a(t)b(s)ds + \int_0^{\beta(t)} a(t)b(s)ds$$

$$\leq 2\int_0^{\beta(t)} a(t)b(s)ds$$

3.7 $\leq 2\vartheta(t)$

Hence, in virtue of hypothesis(H_5), we infer that there exists $T > 0$ such that $\vartheta(t) \leq \epsilon$ for $t \geq T$. Thus, for $t \geq T$ from (3.7), we obtain that

$$|Bx(t) - By(t)| \leq 2\epsilon$$

Furthermore, let us assume that $t \in [0,T]$.

Then evaluating similarly as above we obtain the estimate:

$$|Bx(t) - By(t)| \leq \int_0^{\beta(t)} |g(s,x_s) - g(s,y_s)|ds$$

$$\leq \int_0^{\beta(t)} \omega_r^T(g,\epsilon)ds$$

3.8 $\leq \beta_T \omega_r^T(g,\epsilon)$

Where we denoted $\beta_T = sup\{\beta(t: t \in [0,T])\}$, and

$$\omega_r^T(g,\epsilon) = sup\{|g(s,x_s) - g(s,y_s)|: s \in [0,T], x, y \in [-r,r], |x - y| \leq \epsilon\}.$$

Obviously, we have in view of continuity of β that $\beta_T < \infty$. Moreover; from the uniform continuity of the function $g(t,x_t)$ on the set $I \times C$, we derive that $\omega_r^T(g,\epsilon) \to 0$ as $\epsilon \to 0$. Now, linking (3.7),(3.8) and the above fact we conclude that the operator B maps continuously the ball $\overline{\mathcal{B}}_r(0)$ into itself.

Next, we show that B is compact on $\overline{\mathcal{B}}_r(0)$. To finish, it is enough to show that every sequence $\{Bx_n\}$ in $B(\overline{\mathcal{B}}_r(0))$ has Cauchy subsequence. Now by (H_4) and(H_5),

3.9 $|Bx_n(t)| \leq |\phi(0)| + \int_0^{\beta(t)} |g(s,x_s)|ds \leq K_1 + \vartheta(t)$

$$\leq K_1 + K_2$$

For all $t \in I$. Taking supremum over t, we obtain $\|Bx_n\| \leq K_1 + K_2$ for all $n \in N$. This shows that $\{Bx_n\}$ is a uniformly bounded sequence in $B(\overline{\mathcal{B}}_r(0))$. We show that it is also equicontinuous.

Let $\epsilon > 0$ be given, since $\lim_{t\to\infty}\phi(t) = 0$ and $\lim_{t\to\infty}\vartheta(t) = 0$ there are constants $T_1 > 0$ and $T_2 > 0$ such that $|\phi(t)| < \epsilon/2$ for all $t \geq T_1$ and $|\vartheta(t)| < \epsilon/2$ for all $t \geq T_2$. Let $T = max\{T_1, T_2\}$. Let $t, \lambda \in J$ be arbitrary. If $t, \lambda \in J$ then we have

$$|Bx_n(t) - Bx_n(\lambda)| \leq |\phi(0) - \phi(0)|$$

$$+\left|\int_0^{\beta(t)} g(s, x_{s_n})ds - \int_0^{\beta(\lambda)} g(s, x_{s_n})ds\right|$$

$$\leq \left|\int_{\beta(\lambda)}^{\beta(t)} g(s, x_{s_n})ds\right| \quad \text{for } t \in I.$$

And $|Bx_n(t) - Bx_n(\lambda)| \leq |\phi(t) - \phi(\lambda)|$ for $t \in I_0$.

By the uniform continuity of the function ϕ and J and the function g in $I \times C$, we obtain

$$|Bx_n(t) - Bx_n(\lambda)| \to 0 \text{ as } t \to \lambda.$$

Hence $\{Bx_n\}$ is an equicontinuous sequence of functions in X. Now an application of Arzela-Ascoli theorem yields that $\{Bx_n\}$ has a uniformly convergent subsequence on the compact subset $[0, T]$ of $\mathbb{R}$. Without loss of generality, call the $\{Bx_n\}$ is Cauchy in X.

Now $|Bx_n(t) - Bx(t)| \to 0$ as $x \to \infty$ for all $\in [0, T]$.

Then for given $\epsilon > 0 \ \exists\ x_0 \in N$ such that

$$\sup_{0 \leq p \leq T} \int_0^{\beta(p)} |g(s, x_{s_m}) - g(s, x_{s_n})| ds < \epsilon/2$$

For all $m, n \geq n_0$. Therefore, if $m, n \geq n_0$ then we have

$$\|Bx_m - Bx_n\| = \sup_{0 \leq t \leq \infty} \left|\int_0^{\beta(t)} |g(s, x_{s_m}) - g(s, x_{s_n})| ds\right|$$

$$\leq \sup_{0\leq p\leq T} \left|\int_0^{\beta(t)} |g(s, x_{s_m}) - g(s, x_{s_n})| ds\right|$$

$$+ \sup_{p\leq T} \int_0^{\beta(p)} [|g(s, x_{s_n})| + g(s, x_{s_n})] ds < \epsilon$$

This shows that $\{Bx_n\} \subset B(\overline{\mathcal{B}}_r(0)) \subset X$ is Cauchy. Since X is complete, $\{Bx_n\}$ converges to a point in X. As $B(\overline{\mathcal{B}}_r(0))$ is closed $\{Bx_n\}$ converges to a point in $B(\overline{\mathcal{B}}_r(0))$. Hence $B(\overline{\mathcal{B}}_r(0))$ is relatively compact and consequently B is continuous and compact operator on $\overline{\mathcal{B}}_r(0)$.

Next, we show that $AxBx \in \overline{\mathcal{B}}_r(0)$, for all $x \in \overline{\mathcal{B}}_r(0)$. Let $x \in \overline{\mathcal{B}}_r(0)$ be arbitrary. Then

$$|Ax(t)Bx(t)| \leq |Ax(t)||Bx(t)|$$

$$\leq |f(t, x(t))| \left(|\phi(0)| + \int_0^{\beta(t)} |g(s, x_s)| ds\right)$$

$$\leq [|f(t, x(t)) - f(t, 0)| + |f(t, 0)|] \left(|\phi(0)| + at0\beta tbsds\right.$$

$$\leq [\ell(t)|x(t)| + F(t)](|\phi(0)| + \vartheta(t))$$

$$\leq [L|x(t)| + F_0](K_1 + K_2)$$

$$\leq L(K_1 + K_2)\|x\| + F_0(K_1 + K_2)$$

$$= \frac{F_0(K_1+K_2)}{1-L(K_1+K_2)} = r \quad \text{for all } t \in \mathbb{R}_+.$$

Taking the supremum over t, we obtain $\|AxBx\| \leq r$, for all $x \in \overline{\mathcal{B}}_r(0)$. Hence hypothesis (c) of theorem (2.9) holds. Here, one has

$$M = \|B(\overline{\mathcal{B}}_r(0))\| = sup\{\|Bx\|: x \in \overline{\mathcal{B}}_r(0)\}$$

$$= sup\left\{\sup_{t\geq 0}\left\{|\phi(0)| + \int_0^{\beta(t)} |g(s, x_s)| ds\right\}: x \in \overline{\mathcal{B}}_r(0)\right\}$$

$$\leq \sup_{t\geq 0} |\phi(t)| + \sup_{t\geq 0} \vartheta(t). \ \leq K_1 + K_2$$

And therefore, $M_K = L(K_1 + K_2) < \infty$. Now we apply Theorem (2.9) to conclude that the FDE (1.1) has a solution on J.

Finally, we show the uniform locally asymptotic attractivity of the solution for FDE (1.1). Let x, y be any two solutions of the FDE in $\overline{\mathcal{B}}_r(0)$ defined on I. Then we have

$$|x(t) - y(t)| \leq \left|f(t, x(t))\left(\phi(0) + \int_0^{\beta(t)} g(s, x_s) ds\right)\right|$$

$$+ \left|f(t, y(t))\left(\phi(0) + \int_0^{\beta(t)} g(s, y_s) ds\right)\right|$$

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$$\leq |f(t,x(t))|\left(|\phi(0)| + \int_0^{\beta(t)} |g(s,x_s)|ds\right) + |f(t,y(t))|\left(|\phi(0)| + \int_0^{\beta(t)} |g(s,y_s)|ds\right)$$

$$\leq 2(Lr + F_0)(|\phi(0)| + \vartheta(t)) \quad \text{for all } t \in I.$$

Since $\lim_{t\to\infty} \vartheta(t) = 0$, for $\in > 0$, there are real numbers $T' > 0$ and $T'' > 0$ such that for all $t \geq T' \, |\phi(0)| < \frac{\epsilon}{2(Lr+F_0)}$ and $\vartheta(t) < \frac{\epsilon}{2(Lr+F_0)}$ for all $t \geq T''$. If we choose $T^* = max\{T', T''\}$, then from the above inequality it follows that $|x(t) - y(t)| \leq \epsilon$ for all $t \in T^*$.

It is easy to prove that every solution of the FDE (1.1) is asymptotic to zero on I. Consequently, the FDE (1.1) has a solution and all the solutions are uniformly locally asymptotically attractive on I. The proof is complete.

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