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APPROXIMATING COMMON FIXED POINTS OF THREE ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT

In this paper we have studied necessary and sufficient conditions for convergence of Ishikawa-type iterative sequences to a common fixed point of the mappings defined on a nonempty closed convex subset of a Banach space. Sequences we have considered involving three asymptotically quasi-non expansive mappings

Keywords : Asymptotically non expansive mapping , convex function , real Banach space , closed convex subset

1 Introduction

The mapping T is called non expansive if $|| Tx - Ty || \le || x - y ||$ for all $x, y \in K$; and is called quasi-non expansive if $F(T) \ne \Phi$ and

$$|| Tx - p || \le || x - p || \text{ for all } x \in K \text{ and } p \in F(T).$$

It is therefore clear that a non expansive mapping with a nonempty fixed point set is quasi-non expansive and an asymptotically non expansive mapping with a nonempty fixed point set is asymptotically quasi-non expansive. The converses do not hold in general.

2 Preliminaries

Lemma 2.1 : Let E be a uniformly convex Banach space and $\{\alpha_n\}$ a sequence in $[\varepsilon, 1 - \varepsilon]$ for some $\varepsilon \in (0, 1)$. Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in E such that

$\limsup_{n\to\infty} || x_n || \le r$, $\limsup_{n\to\infty} || y_n || \le r$, and $\limsup_{n\to\infty} || \alpha_n x_n + (1 - \alpha_n) y_n || = r$ hold for some $r \ge 0$. Then $\lim_{n\to\infty} || x_n - y_n || = 0$.

Lemma 2.2 : Let $p > 1$ and $R > 1$ be two fixed numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$|| \lambda x + (1 - \lambda) y ||^p \le \lambda || x ||^p + (1 - \lambda) || y ||^p - W_p(\lambda)\, g(|| x - y ||)$ for all $x, y \in B_R(0) = \{x \in E : || x || \le R\}$, and $\lambda \in [0,1]$, where $W_p(\lambda) = \lambda (1 - \lambda)^p + \lambda^p (1 - \lambda)$.

Lemma 2.3 : Let $\{\lambda_n\}$ and $\{\sigma_n\}$ be sequences of non-negative real numbers such that $\lambda_{n+1} \le \lambda_n + \sigma_n$, $\forall\, n \ge 1$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$. Then $\lim_{n\to\infty} \lambda_n$ exists. Moreover, if there exists a subsequence $\{\lambda_{nj}\}$ of $\{\lambda_n\}$ such that $\lambda_{nj} \to 0$ as $j \to \infty$, then $\lambda_n \to 0$ ass $n \to \infty$.

3 Main Results

Theorem 3.1. Let E be a real Banach space and K a nonempty closed convex subset of E. Let $R, S, T : K \to K$ be three asymptotically quasi-non expansive mappings with sequences $\{u_n\} . \{v_n\} . \{w_n\} \subset [0, \infty)$ such that

$$\sum_{n=1}^{\infty} u_n < \infty \text{ and } \sum_{n=1}^{\infty} v_n < \infty; \sum_{n=1}^{\infty} w_n < \infty \text{ and}$$

$$F = F(S) \cap F(T) \cap F(R) \ne \phi.$$

Let $\{\alpha_n\}, \{\beta_n\} \{\gamma_n\}$ be sequences in $[0, 1]$. $\{x_n\}$ be defined as

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n S^n y_n$$
$$y_n = (1 - \beta_n) x_n + \beta_n T^n z_n$$
$$z_n = (1 - \gamma_n) x_n + \gamma_n R^n x_n$$

then

(1) $|| x_{n+1} - p || \le (1 + b_n) || x_n - p ||$.

(2) There exists a constant M > 0 such that

$$|| x_{n+m} - p || \le M || x_n - p ||.$$

For all n, $m \ge 1$ and $p \in F$.

Proof : Let $p \in F$;

$$\begin{aligned} || z_n - p || &= || (1 - \gamma_n) x_n + \gamma_n R^n x_n - p || \\ &\le (1 - \gamma_n) || x_n - p || + \gamma_n (1 + w_n) \\ &\qquad || x_n - p || \end{aligned}$$

$$\begin{aligned} || x_{n+1} - p || &= || (1 - \alpha_n) x_n + \alpha_n S^n y_n - p || \\ &\le (1 - \alpha_n) || x_n - p || + \alpha_n (1 + u_n) \\ &\qquad || y_n - p || \end{aligned}$$

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$$\begin{aligned} \| y_n - p \| &= \| (1-\beta_n) x_n + \beta_n T^n z_n - p \| \\ &= (1-\beta_n) \| x_n - p \| + \beta_n (1+v_n) \| z_n - p \| \end{aligned}$$

by these equations,

$$\begin{aligned} \| x_{n+1} - p \| &\le (1-\alpha_n) \| x_n - p \| + \alpha_n (1+u_n) [\| (1-\beta_n) \| x_n - p \| + \beta_n (1+v_n) \| z_n - p \|] \\ &= (1-\alpha_n) \| x_n - p \| + \alpha_n (1+u_n)(1-\beta_n) \| x_n - p \| + \alpha_n\beta_n (1+u_n)(1+v_n) [(1-\gamma_n) \| x_n - p \| + \gamma_n (1+w_n) \| x_n - p \|] \\ &= (1-\alpha_n) \| x_n - p \| + \alpha_n (1-\beta_n + u_n - u_n\beta_n) \| x_n - p \| + \alpha_n\beta_n (1+u_n)(1+v_n)(1-\gamma_n) \| x_n - p \| + \alpha_n\beta_n\gamma_n (1+u_n)(1+v_n)(1+w_n) \| x_n - p \| \\ \| x_{n+1} - p \| &\le (1+b_n) \| x_n - p \| \end{aligned}$$

Where

$b_n = (u_n + v_n + w_n + u_nv_n + u_nw_n + v_nw_n + v_nv_nw_n)$.with $\sum_{n=1}^{\infty} b_n < \infty$

(2) For any $n, m \ge 1$

$$\begin{aligned} \| x_{n+m} - p \| &\le (1 + b_{n+m-1}) \| x_{n+m-1} - p \| \\ &\le \exp (b_{n+m-1}) \| x_{n+m-1} - p \| \\ &\le \ldots\ldots \le \exp \left(\sum_{k=n}^{n+m-1} b_k \right) \| x_n - p \| \end{aligned}$$

Let $M = \exp \left(\sum_{k=1}^{\infty} b_k \right)$. Then $0 < M < \infty$ and

$$\| x_{n+m} - x^* \| \le M \| x_n - p \|.$$

Theorem 3.2. Let E be a real Banach space and K a nonempty closed convex subset of E. Let $R, S, T : K \to K$ be three asymptotically quasi-non expansive mappings with sequences $\{u_n\} . \{v_n\} . \{w_n\} \subset [0, \infty)$ such that

$$\sum_{n=1}^{\infty} u_n < \infty \text{ and } \sum_{n=1}^{\infty} v_n < \infty; \sum_{n=1}^{\infty} w_n < \infty \text{ and } F = F(S) \cap F(T) \cap F(R) \neq \phi.$$

Let $\{\alpha_n\}, \{\beta_n\} \{\gamma_n\}$ be sequences in $[0, 1]$. $\{x_n\}$ be defined as

$$\begin{aligned} x_{n+1} &= (1-\alpha_n) x_n + \alpha_n S^n y_n \\ y_n &= (1-\beta_n) x_n + \beta_n T^n z_n \\ z_n &= (1-\gamma_n) x_n + \gamma_n R^n x_n \end{aligned}$$

then

$$\lim_{n\to\infty} \| S^n x_n - x_n \| = \lim_{n\to\infty} \| T^n x_n - x_n \| = \lim_{n\to\infty} \| R^n x_n - x_n \| = 0$$

Proof : Let $p \in F$ then by theorem4.1 and Lemma 2.3;

$\lim_{n\to\infty} \| x_n - p \|$ exists. Let this limit be r. If $r = 0$; result is obvious by the continuity of R, T, S so let $r > 0$.

We claim

$$\lim_{n\to\infty} \| S^n x_n - x_n \| = \lim_{n\to\infty} \| T^n x_n - x_n \| = \lim_{n\to\infty} \| R^n x_n - x_n \| = 0$$

Since $\{x_n\}$ is bounded; $\exists\, R > 0$; such that $x_n - p, y_n - p \in B_R(0)$ for all $n \ge 1$. Now

by Lemma 2.2; we have

$$\begin{aligned} \| z_n - p \|^2 &= \| (1-\gamma_n) x_n + \gamma_n R^n x_n - p \|^2 \\ &\le \gamma \| R^n x_n - p \|^2 + (1-\gamma_n) \| x_n - p \|^2 - W_2(\gamma_n)\, g(\| R^n x_n - x_n \|) \\ &\le \gamma_n (1+w_n)^2 \| x_n - p \|^2 + (1-\gamma_n) \| x_n - p \|^2 \\ &\le (1+w_n)^2 \| x_n - p \| \end{aligned}$$

$$\begin{aligned} \| y_n - p \|^2 &= \| [(1-\beta_n) x_n + \beta_n T^n z_n] - p \|^2 \\ &\le \beta_n \| T^n z_n - p \|^2 + (1-\beta_n) \| x_n - p \|^2 - W_2(\beta_n)\, g(\| T^n z_n - x_n \|) \\ &\le \beta_n (1+v_n)^2 \| z_n - p \|^2 + (1-\beta_n) \| x_n - p \|^2 - W_2(\beta_n)\, g(\| T^n z_n - p \|^2) \\ &\le (1+v_n)^2 \| z_n - p \|^2 \end{aligned}$$

From Lemma 2.2

$$\begin{aligned} \| x_{n+1} - p \|^2 &= \| (1-\alpha_n) x_n + \alpha_n S^n y_n - p \|^2 \\ &\le (1-\alpha_n) \| x_n - p \|^2 + \alpha_n (1+u_n)^2 \| y_n - p \|^2 - W_2(\alpha_n)\, g(\| S^n y_n - x_n \|) \\ &\le (1-\alpha_n) \| x_n - p \|^2 + \alpha_n (1+u_n)^2 (1+v_n)^2 (1+w_n)^2 \| x_n - p \|^2 - W_2(\alpha_n)\, g(\| S^n y_n - x_n \|) \\ &\le \| x_n - p \|^2 + C_n A^2 - W_2(\alpha_n)\, g(\| S^n y_n - x_n \|), \end{aligned}$$

clearly $W_2(\alpha_n) \ge \varepsilon^2$ and $\sum_{n=1}^{\infty} C_n < \infty$

so $\varepsilon^2 \sum_{n=1}^{\infty} g(\| S^n y_n - x_n \|) < \| x_n - p \|^2 + A^2 \sum_{n=1}^{\infty} C_n < \infty$

i.e. $\lim_{n\to\infty} g(\| S^n y_n - x_n \|) = 0.$

$\lim_{n\to\infty} \| S^n y_n - x_n \| = 0$ as g is strictly increasing and continuous at 0.

Since S is asymptotically quasi-non-expansive,

$$\begin{aligned} \| x_n - p \| &\le \| x_n - S^n y_n \| + \| S^n y_n - p \| \\ &\le \| x_n - S^n y_n \| + (1+u_n) \| y_n - p \| \end{aligned}$$

i.e $.r \le \lim \| y_n - p \|.$

Also,.

$$\begin{aligned} \| y_n - p \| &\le \| (1-\beta_n) x_n + \beta_n T^n z_n - p \| \\ &\le (1-\beta_n) \| x_n - p \| + \beta_n (T^n z_n - T^n x_n + T^n x_n - p). \end{aligned}$$

$\limsup_{n\to\infty} || y_n - p || \le r$. Therefore,

$\lim_{n\to\infty} || y_n - p || = r$ and so

$$|| \beta_n (T^n x_n - p) + (1 - \beta_n)(x_n - p) || = r.$$

We have, $\limsup_{n\to\infty} || T^n x_n - p || \le r$, it follows from Lemma 3.1 that

$$\lim_{n\to\infty} || T^n x_n - x_n || = 0.$$

Also, we have

$$|| S^n x_n - x_n || \le || S^n x_n - S^n y_n || + || S^n y_n - x_n ||.$$

Since S is uniformly continuous and

$|| x_n - y_n || \to 0$ as $n \to \infty$, it follows from above inequality that $\lim_{n\to\infty} || S^n x_n - x_n || = 0$.

Simlarly $\lim_{n\to\infty} || R^n x_n - x_n || = 0$.

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