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(0,2) - INTERPOLATION ON THE UNIT CIRCLE

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ABSTRACT:
In this paper,we consider (0, 2)- interpolation on the nodes , which are vertically projected zeros of the $(1-x^2)P_n^{(\alpha,\beta)}(x)$on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial . We obtain the explicit forms and establish a convergence theorem for the interpolatory polynomial.

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KEYWORDS: Jacobi polynomial, Explicit representation , Convergence,(0,2)- interpolation, Lacunary interpolation.

§1. INTRODUCTION

P.Turan [14] initiated the study of (0,2)- interpolation in order to get approximate solution of the differential equation:

$$y'' + fy = 0$$

In 1975,L.G. Pal [4] introduced a modification of Hermite-Fejer interpolation in which the function values and the first derivatives were prescribed on two set of nodes .In 1983,L.Szilli [10] studied the problem in which first derivatives are interpolated at the zeroes of P_n and the function values are interpolated at the zeroes of P_n' Further, J. Balazs [2] generalized the results of P.Turan's problem.

In 1960, O. Kis [3] initiated the type of problem for the special knots $Z_n = \left\{ z_k = e^{\frac{2\Pi ik}{n}}, k = 1(1)n \right\}$, the nth roots of unity.He showed that (0,1, …….., r – 2, r)-interpolation on Z_n is regular. Later, Sharma [5] [6] extended the results to (o,m) and $(0, m_1, m_2)$ cases. Further, Sharma and his associates [7] considered regularity, explicit representation and convergence problem of $(m_0, m_1, \ldots\ldots\ldots, m_q)$interpolation on unit circle. In 1995, S.Xie [15] considered the regularity of (0,1,……….r - 2, r)*- interpolation and (0, 1,………..,r– 2, r) interpolation for the vertically projected zeroes of Jacobi polynomial . Also, S. Bahadur and K.K.Mathur[1] considered the weighted (0,2)-interpolation and obtained the explicit forms and convergence for the same.

In this paper, we consider (0,2)- interpolation on the vertically projected zeroes of $(1-x^2)P_n^{(\alpha,\beta)}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial. We obtain the explicit forms and establish a convergence theorem for the interpolatory polynomial. In section 2,we give some preliminaries and in section 3 ,we describe the problem and obtained the regularity of the same. In section 4, we give the explicit formulae of the interpolatory polynomials. In sections 5 and 6, estimation and convergence of interpolatory polynomials are considered respectively.

§2. PRELIMINARIES

In this section, we shall give some well-known results, which we shall use.

The differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ is

$$(2.1)\quad (1-x^2)P_n^{(\alpha,\beta)''}(x)+[\beta-\alpha-(\alpha+\beta+2)x]P_n^{(\alpha,\beta)'}(x)+n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0$$

$$(2.2)\quad W(z)=\prod_{k=1}^{2n}(z-z_k)=K_nP_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right)z^n$$

$$(2.3)\quad R(z)=(z^2-1)W(z)$$

We shall require the fundamental polynomials of Lagrange interpolation based on the nodes as zeroes of R(z) and W(z) are given by:

$$(2.4)\quad L_k(z)=\frac{R(z)}{R'(z_k)(z-z_k)},k=0(1)2n+1$$

$$(2.5)\quad L_{1k}(z)=\frac{W(z)}{W'(z_k)(z-z_k)},k=1(1)2n$$

We will also use the following results

$$(2.6)\quad (-1)^nW'(z_{n+k})=W'(z_k)=-\frac{1}{2}K_nP_n^{(\alpha,\beta)'}(x_k)(1-z_k^2)z_k^{n-2},k=1(1)n$$

$$(2.7)\quad (-1)^{n-1}W''(z_{n+k})=W''(z_k)=-\frac{1}{2}K_nP_n^{(\alpha,\beta)'}(x_k)\left[\begin{matrix}-\begin{pmatrix}2(\beta-\alpha)z_k\\-(\alpha+\beta+2)(1+z_k^2)\end{pmatrix}\\+2n(1-z_k^2)-2\end{matrix}\right]z_k^{n-3},k=1(1)n$$

$$(2.8)\quad R'(z_k)=(z_k^2-1)W'(z_k)$$

$$(2.9)\quad R''(z_k)=W'(z_k)\left[4z_k^2+2(\beta-\alpha)z_k-(\alpha+\beta+2)(1+z_k^2)-2n(1-z_k^2)+2\right]z_k^{-1}$$

(2.10) $$\frac{R''(z_k)}{R'(z_k)} = \frac{\left[4z_k^2 + (\beta-\alpha)2z_k - (\alpha+\beta+2)(1+z_k^2) - 2n(1-z_k^2) + 2\right]z_k^{-1}}{(z_k^2-1)}$$

We will also use the following well known inequalities (see [9])

(2.11) $(1-x^2)^{\frac{1}{2}} P_n^{(\alpha,\beta)}(x) = o(n^{\alpha-1})$ for $\alpha > 0$, $x \in [-1,1]$

(2.12) $(1-x^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2}$

(2.13) $\left|P_n^{(\alpha,\beta)'}(x_k)\right| \sim k^{-\alpha-\frac{3}{2}} n^{\alpha+2}$

(2.14) $\left|P_n^{(\alpha,\beta)}(x)\right| = o(n^{\alpha}), \alpha > 0$

(2.15) $\left|P_n^{(\alpha,\beta)'}(x)\right| = o(n^{\alpha+2})$

Also,

(2.16) $J_{1j}(z) = \int_0^z (t-1)^{\frac{1-2\alpha}{2}} (t+1)^{\frac{1-2\beta}{2}} t^{\frac{2n+\alpha+\beta}{2}+j-1} W(t)\,dt$, $j = 0,1$

And further, we have,

(2.17) $J_{1j}(-1) = (-1)^{\frac{-\alpha-\beta}{2}+j} J_{1j}(1)$

§3. THE PROBLEM AND REGULARITY: Let $Z_n = \{z_k : k = 0(1)2n+1\}$ satisfying:

(3.1) $$Z_n = \{z_0 = 1, z_{2n+1} = -1,$$
$$z_k = \cos\theta_k + i\sin\theta_k, z_{n+k} = -z_k, k = 1(1)n\}$$

where $\{x_k = \cos\theta_k : k = 1(1)n\}$ are the zeroes of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x), 1 > x_1 > x_2 .... > x_n > -1.$
Here, we are interested in determining the interpolatory polynomials $R_n(z)$ of degree $\leq 4n+3$ satisfying the following conditions:

(3.2) $$\begin{cases} R_n(z_k) = \alpha_k, k = 0(1)2n+1 \\ R''_n(z_k) = \beta_k, k = 0(1)2n+1 \end{cases}$$

where α_k and β_k are arbitrary given complex numbers. Here, we are interested in establishing a convergence theorem for the same.

THEOREM 1: (0,2)-interpolation is regular on Z_n

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PROOF: It is sufficient, if we show the unique solution of (3.2) is $R_n(z)\equiv 0$, when all data $\alpha_k=\beta_k=0$. In this case, we have $R_n(z)=R(z)q(z)$,

where q (z) is a polynomial of degree ≤ 2n +1.

By $R_n(z_k)\equiv 0$ and (2.10) , we obtain

$$2z_k\left(z_k^2-1\right)q'(z_k)+\left[4z_k^2+2(\beta-\alpha)z_k-(\alpha+\beta+2)\left(1+z_k^2\right)-2n\left(1-z_k^2\right)+2\right]q(z_k)=0$$

We can suppose constants a and b such that,

(3.3) $$2z\left(z^2-1\right)q'(z)+\left[4z^2+2(\beta-\alpha)z-(\alpha+\beta+2)\left(1+z^2\right)-2n\left(1-z^2\right)+2\right]q(z)=(az+b)R(z)$$

Solving (3.3), and using (2.16) , we get

$$\begin{cases} aJ_{11}(1)+bJ_{10}(1)=0 \\ aJ_{11}(-1)+bJ_{10}(-1)=0 \end{cases}$$

Using (2.17),we get a=b=0.

$\Rightarrow R_n(z)\equiv q(z)\equiv 0.$

Hence, the theorem follows.

§4. EXPLICIT REPRESENTATION OF INTERPOLATORY POLYNOMIALS

We shall write $R_n(z)$ satisfying (3.2) as:

(4.1) $$R_n(z)=\sum_{k=0}^{2n+1}\alpha_k A_k(z)+\sum_{k=0}^{2n+1}\beta_k B_k(z)$$

Where $A_k(z)$ and $B_k(z)$ are unique polynomials, each of degree at most 4n+3 satisfying the conditions:

(4.2) $$\begin{cases} A_k(z_j)=\delta_{jk}, & j,k=0(1)2n+1 \\ A_k''(z_j)=0, & j,k=0(1)2n+1 \end{cases}$$

(4.3) $$\begin{cases} B_k(z_j)=0, & j,k=0(1)2n+1 \\ B_k''(z_j)=\delta_{jk}, & j,k=0(1)2n+1 \end{cases}$$

THEOREM 2: For k=0(1)2n+1,we have,

(4.4) $$B_k(z)=b_k^* z^{\frac{-2n-\alpha-\beta}{2}}(z-1)^{\frac{2\alpha-1}{2}}(z+1)^{\frac{2\beta-1}{2}}R(z)J_k(z)$$

where,

(4.5) $$J_k(z)=\int_0^z t^{\frac{2n+\alpha+\beta}{2}-1}(t-1)^{\frac{1-2\alpha}{2}}(t+1)^{\frac{1-2\beta}{2}}L_k(t)\,dt$$

(4.6) $$b_k^*=\frac{z_k}{2R'(z_k)}$$

THEOREM 3: For k=0(1)2n+1,we have,

(4.7) $$A_k(z) = L_k^2(z) + b_k B_k(z) + z^{\frac{-2n-\alpha-\beta}{2}}(z-1)^{\frac{2\alpha-1}{2}}(z+1)^{\frac{2\beta-1}{2}} R(z) S_k$$

where,

(4.8) $$S_k(z) = -\int_0^z t^{\frac{2n+\alpha+\beta}{2}}(t-1)^{\frac{1-2\alpha}{2}}(t+1)^{\frac{1-2\beta}{2}} \frac{[L_k'(t) - L_k'(z_k)L_k(t)]}{(t-z_k)R'(z_k)} dt$$

(4.9) $$b_k = -4L_k'^2(z_k)$$

§5. ESTIMATION OF FUNDAMENTAL POLYNOMIALS:

LEMMA 1: Let $L_k(z)$ be given by (2.4). Then

(5.1) $$\max_{|z|=1} \sum_{k=0}^{2n+1} |L_k(z)| \le \frac{c}{k^{-\alpha+\frac{3}{2}}}$$

where c is a constant independent of n and z.

LEMMA 2:For $|z| \le 1$j we have

(5.2) $$\sum_{k=0}^{2n+1} |B_k(z)| \le cn^{2\alpha-2} \log n.$$

where $B_k(z)$ is given in theorem 2 and c is a constant independent of n and z.

PROOF: Using lemma 1, (2.11) ,(2.12) and (2.13) ,we get (5.2),

LEMMA 3: For z=$e^{i\theta}(0 \le \theta \le 2\pi)$,we have

(5.3) $$\sum_{k=0}^{2n+1} |S_k(z)| \le c \frac{n^{\alpha+1}}{k^{-2\alpha+2}}$$

where $S_k(z)$ is given in (4.8) and c is a constant independent of n and z.

PROOF: Since

(5.4) $$z^2(z^2-1)W''(z) = [2(\beta-\alpha)z - (\alpha+\beta)(1+z^2) - \{2n(1-z^2)+z^2\}] + n[2(\alpha+\beta)z^2 - 2(\alpha-\beta)z + 2z^2]W(z)$$

(5.5) $$\begin{cases} L_k(z) = \dfrac{R'(z)}{R'(z_k)} - (z-z_k)L_k'(z) \\ L_k'(z) = \dfrac{R''(z)}{2R'(z_k)} - \dfrac{1}{2}(z-z_k)L_k''(z) \end{cases}$$

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Let us suppose that:

$$(5.6)\qquad \mathrm{H}_k(z) = -\int_0^z t^{\frac{2n+\alpha+\beta}{2}}(t-1)^{\frac{1-2\alpha}{2}}(t+1)^{\frac{1-2\beta}{2}}\frac{[L_k'(t)-L_k'(z_k)L_k(t)]}{(t-z_k)}dt$$

Then by using equations (5.4) and (5.5) in (5.6), we get the following equation:

$$(5.7)\qquad \mathrm{H}_k(z) = -\frac{\{n(\alpha+\beta)+(n+1)\}}{(z_k^2-1)}\int_0^z t^{\frac{2n+\alpha+\beta}{2}}(t-1)^{\frac{1-2\alpha}{2}}(t+1)^{\frac{1-2\beta}{2}}L_{1k}(t)dt$$

$$+\frac{n(\alpha-\beta)}{(z_k^2-1)}\int_0^z t^{\frac{2n+\alpha+\beta}{2}}(t-1)^{\frac{1-2\alpha}{2}}(t+1)^{\frac{1-2\beta}{2}}L_{1k}(t)dt$$

$$+\frac{4z_k^2+2(\alpha-\beta)z_k-(\alpha+\beta+2)(1+z_k^2)}{(z_k^2-1)^2}\int_0^z t^{\frac{2n+\alpha+\beta}{2}}(t-1)^{\frac{1-2\alpha}{2}}(t+1)^{\frac{1-2\beta}{2}}L_{1k}(t)dt$$

$$+\frac{\{2n(z_k^2-1)+2\}}{z_k(z_k^2-1)^2}\int_0^z t^{\frac{2n+\alpha+\beta}{2}}(t-1)^{\frac{1-2\alpha}{2}}(t+1)^{\frac{1-2\beta}{2}}L_{1k}(t)dt$$

$$+\frac{\{(\beta-\alpha)z_k-(\alpha+\beta-1)\}}{(z_k^2-1)R'(z_k)}\int_0^z t^{\frac{2n+\alpha+\beta}{2}}(t-1)^{-2\alpha}(t+1)^{-2\beta}W'(t)dt$$

$$+\frac{\{(\alpha+\beta-2)z_k^2+2(\alpha-\beta)z_k+(\alpha+\beta)\}}{2z_k(z_k^2-1)R'(z_k)}\int_0^z t^{\frac{2n+\alpha+\beta}{2}}(t-1)^{\frac{1-2\alpha}{2}}(t+1)^{\frac{1-2\beta}{2}}W'(t)dt$$

$$-\frac{(\alpha+\beta)}{2z_kR'(z_k)}\int_0^z t^{\frac{2n+\alpha+\beta}{2}}(t-1)^{\frac{1-2\alpha}{2}}(t+1)^{\frac{1-2\beta}{2}}W'(t)dt+\frac{n}{z_k}\int_0^z t^{\frac{2n+\alpha+\beta}{2}}(t-1)^{\frac{1-2\alpha}{2}}(t+1)^{\frac{1-2\beta}{2}}\{L_k(t)-(t-z_k)L_k'(t)\}dt$$

$$-\frac{2n}{z_k(z_k^2-1)W'(z_k)}\int_0^z t^{\frac{2n+\alpha+\beta}{2}}(t-1)^{\frac{1-2\alpha}{2}}(t+1)^{\frac{1-2\beta}{2}}W(t)dt+\frac{1}{2}\int_0^z t^{\frac{2n+\alpha+\beta}{2}}(t-1)^{\frac{1-2\alpha}{2}}(t+1)^{\frac{1-2\beta}{2}}L_k''(t)dt$$

$$-L_k'(z_k)\int_0^z t^{\frac{2n+\alpha+\beta}{2}}(t-1)^{\frac{1-2\alpha}{2}}(t+1)^{\frac{1-2\beta}{2}}L_k(t)dt$$

$$=\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5+\mathrm{I}_6+\mathrm{I}_7+\mathrm{I}_8+\mathrm{I}_9+\mathrm{I}_{10}+\mathrm{I}_{11}$$

Here, using (2.11) - (2.15), we get

$$(5.8)\qquad \sum_{k=0}^{2n+1}|\mathrm{H}_k(z)| \le c\frac{n}{k^{-\alpha+\frac{3}{2}}}$$

Further,

(5.9) $\left|R'(z_k)\right| \sim \dfrac{k^{-\alpha+\frac{1}{2}}}{n^{\alpha}}$

Using (5.8) and (5.9) in (4.8) ,we get the required lemma.

(5.10) $\displaystyle\sum_{k=0}^{2n+1}\left|A_k(z)\right| \le cn^{2\alpha}\log n$

where, $A_k(z)$is given in (4.7) and c is a constant independent of n and z.

PROOF: We have,

(5.11) $\left|b_k\right| = \left|4L_k'^2(z_k)\right| \le cn^2$

(5.12) $\left|z^{\frac{-2n-\alpha-\beta}{2}}(z-1)^{\frac{2\alpha-1}{2}}(z+1)^{\frac{2\beta-1}{2}}R(z)S_k\right| \le \dfrac{c}{k^{2-2\alpha}}$

Using lemma 2,lemma3,(5.11) and (5.12) in (4.7) , we get (5.10).

§6. CONVERGENCE:

Let f(z) be analytic for $|z|<1$ and continuous for $|z|\le 1$ and $\omega(f,\delta)$be the modulus of continuity of $f(e^{i\theta})$.

THEOREM 5: Let f (z) be continuous in $|z|\le 1$ and analytic in$|z|<1$. Let the arbitrary number β_k's be such that:

(6.1) $\{\left|\beta_k\right| = o\left(n^2\omega\left(f,n^{-1}\right)\right), k = 0(1)2n+1$

Then $\{R_n\}$be defined by :

(6.2) $R_n(z) = \displaystyle\sum_{k=0}^{2n+1} f(z_k)A_k(z) + \sum_{k=0}^{2n+1}\beta_k B_k(z)$

Satisfies the relation:

(6.3) $\left|R_n(z) - f(z)\right| = o\left(n^{2\alpha}\log n\,\omega\left(f,n^{-1}\right)\right)$,

where, $\omega(f,n^{-1})$ is the modulus of continuity of f(z).

To prove theorem 5,we shall need the following:

REMARK: Let f (z) be continuous in $|z|\le 1$ and analytic in$|z|<1$ .Then there exists a polynomial $F_n(z)$ of degree 2n-2 satisfying Jackson's inequality

(6.4) $\quad |f(z)-F_n(z)| \le cw_2(f,n^{-1}) \quad , \quad z=e^{i\theta}(0\le\theta<2\pi)$

And also an inequality due to O.Kis[3]

(6.5) $\quad |F_n^{(m)}(z)| \le cn^m\omega(f,n^{-1}) \quad , \quad \text{for } m\in I^+$

PROOF: Since $R_n(z)$ be given by (6.2) is a uniquely determined polynomial of degree $\le 4n+3$, the polynomial $F_n(z)$ satisfying (6.4) and (6.5) can be expressed as :

$$F_n(z)=\sum_{k=0}^{2n+1}F_n(z_k)A_k(z)+\sum_{k=0}^{2n+1}F_n''(z_k)B_k(z)$$

Then,
$$\begin{aligned}|R_n(z)-f(z)| &\le |R_n(z)-F_n(z)|+|F_n(z)-f(z)|\\ &\le \sum_{k=0}^{2n+1}|f(z_k)-F_n(z_k)||A_k(z)|+\sum_{k=0}^{2n+1}\{|\beta_k|+F_n''(z_k)\}|B_k(z)|\\ &\quad +|F_n(z)-f(z)|\end{aligned}$$

Using $z=e^{i\theta}(0<\theta\le 2\pi)$, (6.1),(6.4),(6.5) and lemma 2 and 4, we get (6.3).

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