

(0,2) - INTERPOLATION ON THE UNIT CIRCLE

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ABSTRACT:

In this paper, we consider (0, 2)- interpolation on the nodes, which are vertically projected zeros of the $(1-x^2)P_n^{(\alpha,\beta)}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial. We obtain the explicit forms and establish a convergence theorem for the interpolatory polynomial.

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§1. INTRODUCTION

P.Turan [14] initiated the study of (0,2)- interpolation in order to get approximate solution of the differential equation:

$$y'' + fy = 0$$

In 1975, L.G. Pal [4] introduced a modification of Hermite-Fejer interpolation in which the function values and the first derivatives were prescribed on two set of nodes. In 1983, L.Szilli [10] studied the problem in which first derivatives are interpolated at the zeroes of P_n and the function values are interpolated at the zeroes of P'_n . Further, J. Balazs [2] generalized the results of P.Turan's problem.

In 1960, O. Kis [3] initiated the type of problem for the special knots

$$Z_n = \left\{ z_k = e^{\frac{2\pi ik}{n}}, k = 1(1)n \right\}, \text{ the } n\text{th roots of unity. He showed that } (0, 1, \dots, r-2, r)\text{-}$$

interpolation on Z_n is regular. Later, Sharma [5] [6] extended the results to $(0, m)$ and $(0, m_1, m_2)$ cases. Further, Sharma and his associates [7] considered regularity, explicit representation and convergence problem of (m_0, m_1, \dots, m_q) interpolation on unit circle. In 1995, S.Xie [15] considered the regularity of $(0, 1, \dots, r-2, r)^*$ - interpolation and $(0, 1, \dots, r-2, r)$ interpolation for the vertically projected zeroes of Jacobi polynomial. Also, S. Bahadur and K.K.Mathur [1] considered the weighted (0,2)-interpolation and obtained the explicit forms and convergence for the same.

In this paper, we consider (0,2)- interpolation on the vertically projected zeroes of $(1-x^2)P_n^{(\alpha,\beta)}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial. We obtain the explicit forms and establish a convergence theorem for the interpolatory polynomial. In section 2, we give some preliminaries and in section 3, we describe the problem and obtained the regularity of the same. In section 4, we give the explicit formulae of the interpolatory polynomials. In sections 5 and 6, estimation and convergence of interpolatory polynomials are considered respectively.

§2. PRELIMINARIES

In this section, we shall give some well-known results, which we shall use.

The differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ is

$$(2.1) \quad (1-x^2)P_n^{(\alpha,\beta)''}(x) + [\beta - \alpha - (\alpha + \beta + 2)x]P_n^{(\alpha,\beta)'}(x) + n(n + \alpha + \beta + 1)P_n^{(\alpha,\beta)}(x) = 0$$

$$(2.2) \quad W(z) = \prod_{k=1}^{2n} (z - z_k) = K_n P_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right) z^n$$

$$(2.3) \quad R(z) = (z^2 - 1)W(z)$$

We shall require the fundamental polynomials of Lagrange interpolation based on the nodes as zeroes of $R(z)$ and $W(z)$ are given by:

$$(2.4) \quad L_k(z) = \frac{R(z)}{R'(z_k)(z - z_k)}, k = 0(1)2n + 1$$

$$(2.5) \quad L_{1k}(z) = \frac{W(z)}{W'(z_k)(z - z_k)}, k = 1(1)2n$$

We will also use the following results

$$(2.6) \quad (-1)^n W'(z_{n+k}) = W'(z_k) = -\frac{1}{2} K_n P_n^{(\alpha,\beta)'}(x_k) (1 - z_k^2) z_k^{n-2}, k = 1(1)n$$

$$(2.7) \quad (-1)^{n-1} W''(z_{n+k}) = W''(z_k) = -\frac{1}{2} K_n P_n^{(\alpha,\beta)'}(x_k) \left[\begin{array}{l} -\left(\frac{2(\beta - \alpha)z_k}{-(\alpha + \beta + 2)(1 + z_k^2)} \right) \\ + 2n(1 - z_k^2) - 2 \end{array} \right] z_k^{n-3}, k = 1(1)n$$

$$(2.8) \quad R'(z_k) = (z_k^2 - 1)W'(z_k)$$

$$(2.9) \quad R''(z_k) = W'(z_k) [4z_k^2 + 2(\beta - \alpha)z_k - (\alpha + \beta + 2)(1 + z_k^2) - 2n(1 - z_k^2) + 2] z_k^{-1}$$

$$(2.10) \quad \frac{R''(z_k)}{R'(z_k)} = \frac{[4z_k^2 + (\beta - \alpha)2z_k - (\alpha + \beta + 2)(1 + z_k^2) - 2n(1 - z_k^2) + 2]z_k^{-1}}{(z_k^2 - 1)}$$

We will also use the following well known inequalities (see [9])

$$(2.11) \quad (1 - x^2)^{\frac{1}{2}} P_n^{(\alpha, \beta)}(x) = o(n^{\alpha-1}) \text{ for } \alpha > 0, x \in [-1, 1]$$

$$(2.12) \quad (1 - x^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2}$$

$$(2.13) \quad \left| P_n^{(\alpha, \beta)'}(x_k) \right| \sim k^{-\alpha - \frac{3}{2}} n^{\alpha+2}$$

$$(2.14) \quad \left| P_n^{(\alpha, \beta)}(x) \right| = o(n^\alpha), \alpha > 0$$

$$(2.15) \quad \left| P_n^{(\alpha, \beta)'}(x) \right| = o(n^{\alpha+2})$$

Also,

$$(2.16) \quad J_{ij}(z) = \int_0^z (t-1)^{\frac{1-2\alpha}{2}} (t+1)^{\frac{1-2\beta}{2}} t^{\frac{2n+\alpha+\beta}{2} + j-1} W(t) dt, \quad j=0,1$$

And further, we have,

$$(2.17) \quad J_{ij}(-1) = (-1)^{\frac{-\alpha-\beta}{2} + j} J_{ij}(1)$$

§3. THE PROBLEM AND REGULARITY: Let $Z_n = \{z_k : k = 0(1)2n + 1\}$ satisfying:

$$(3.1) \quad Z_n = \{z_0 = 1, z_{2n+1} = -1, z_k = \cos \theta_k + i \sin \theta_k, z_{n+k} = -z_k, k = 1(1)n\}$$

where $\{x_k = \cos \theta_k : k = 1(1)n\}$ are the zeroes of the Jacobi polynomial $P_n^{(\alpha, \beta)}(x), 1 > x_1 > x_2 > \dots > x_n > -1$.

Here, we are interested in determining the interpolatory polynomials $R_n(z)$ of degree $\leq 4n + 3$ satisfying the following conditions :

$$(3.2) \quad \begin{cases} R_n(z_k) = \alpha_k, k = 0(1)2n + 1 \\ R_n''(z_k) = \beta_k, k = 0(1)2n + 1 \end{cases}$$

where α_k and β_k are arbitrary given complex numbers. Here, we are interested in establishing a convergence theorem for the same.

THEOREM 1: (0,2)-interpolation is regular on Z_n

PROOF: It is sufficient, if we show the unique solution of (3.2) is $R_n(z) \equiv 0$, when all data $\alpha_k = \beta_k = 0$. In this case, we have $R_n(z) = R(z)q(z)$,

where $q(z)$ is a polynomial of degree $\leq 2n+1$.

By $R_n(z_k) \equiv 0$ and (2.10), we obtain

$$2z_k(z_k^2 - 1)q'(z_k) + [4z_k^2 + 2(\beta - \alpha)z_k - (\alpha + \beta + 2)(1 + z_k^2) - 2n(1 - z_k^2) + 2]q(z_k) = 0$$

We can suppose constants a and b such that,

$$(3.3) \quad 2z(z^2 - 1)q'(z) + [4z^2 + 2(\beta - \alpha)z - (\alpha + \beta + 2)(1 + z^2) - 2n(1 - z^2) + 2]q(z) = (az + b)R(z)$$

Solving (3.3), and using (2.16), we get

$$\begin{cases} aJ_{11}(1) + bJ_{10}(1) = 0 \\ aJ_{11}(-1) + bJ_{10}(-1) = 0 \end{cases}$$

Using (2.17), we get $a=b=0$.

$$\Rightarrow R_n(z) \equiv q(z) \equiv 0.$$

Hence, the theorem follows.

§4. EXPLICIT REPRESENTATION OF INTERPOLATORY POLYNOMIALS

We shall write $R_n(z)$ satisfying (3.2) as:

$$(4.1) \quad R_n(z) = \sum_{k=0}^{2n+1} \alpha_k A_k(z) + \sum_{k=0}^{2n+1} \beta_k B_k(z)$$

Where $A_k(z)$ and $B_k(z)$ are unique polynomials, each of degree at most $4n+3$ satisfying the conditions:

$$(4.2) \quad \begin{cases} A_k(z_j) = \delta_{jk}, \quad j, k = 0(1)2n+1 \\ A_k''(z_j) = 0, \quad j, k = 0(1)2n+1 \end{cases}$$

$$(4.3) \quad \begin{cases} B_k(z_j) = 0, \quad j, k = 0(1)2n+1 \\ B_k''(z_j) = \delta_{jk}, \quad j, k = 0(1)2n+1 \end{cases}$$

THEOREM 2: For $k=0(1)2n+1$, we have,

$$(4.4) \quad B_k(z) = b_k^* z^{\frac{-2n-\alpha-\beta}{2}} (z-1)^{\frac{2\alpha-1}{2}} (z+1)^{\frac{2\beta-1}{2}} R(z) J_k(z)$$

where,

$$(4.5) \quad J_k(z) = \int_0^z t^{\frac{2n+\alpha+\beta}{2}-1} (t-1)^{\frac{1-2\alpha}{2}} (t+1)^{\frac{1-2\beta}{2}} L_k(t) dt$$

$$(4.6) \quad b_k^* = \frac{z_k}{2R'(z_k)}$$

THEOREM 3: For $k=0(1)2n+1$, we have,

$$(4.7) \quad A_k(z) = L_k^2(z) + b_k B_k(z) + z^{\frac{-2n-\alpha-\beta}{2}} (z-1)^{\frac{2\alpha-1}{2}} (z+1)^{\frac{2\beta-1}{2}} R(z) S_k$$

where,

$$(4.8) \quad S_k(z) = -\int_0^z t^{\frac{2n+\alpha+\beta}{2}} (t-1)^{\frac{1-2\alpha}{2}} (t+1)^{\frac{1-2\beta}{2}} \frac{[L'_k(t) - L'_k(z_k)L'_k(t)]}{(t-z_k)R'(z_k)} dt$$

$$(4.9) \quad b_k = -4L_k^2(z_k)$$

§5. ESTIMATION OF FUNDAMENTAL POLYNOMIALS:

LEMMA 1: Let $L_k(z)$ be given by (2.4). Then

$$(5.1) \quad \max_{|z|=1} \sum_{k=0}^{2n+1} |L_k(z)| \leq \frac{c}{k^{-\alpha+\frac{3}{2}}}$$

where c is a constant independent of n and z .

LEMMA 2: For $|z| \leq 1$ we have

$$(5.2) \quad \sum_{k=0}^{2n+1} |B_k(z)| \leq cn^{2\alpha-2} \log n.$$

where $B_k(z)$ is given in theorem 2 and c is a constant independent of n and z .

PROOF: Using lemma 1, (2.11), (2.12) and (2.13), we get (5.2),

LEMMA 3: For $z = e^{i\theta}$ ($0 \leq \theta \leq 2\pi$), we have

$$(5.3) \quad \sum_{k=0}^{2n+1} |S_k(z)| \leq c \frac{n^{\alpha+1}}{k^{-2\alpha+2}}$$

where $S_k(z)$ is given in (4.8) and c is a constant independent of n and z .

PROOF: Since

$$(5.4) \quad z^2(z^2-1)W''(z) = [2(\beta-\alpha)z - (\alpha+\beta)(1+z^2) - \{2n(1-z^2) + z^2\}] + n[2(\alpha+\beta)z^2 - 2(\alpha-\beta)z + 2z^2]W(z)$$

$$(5.5) \quad \begin{cases} L_k(z) = \frac{R'(z)}{R'(z_k)} - (z-z_k)L'_k(z) \\ L'_k(z) = \frac{R''(z)}{2R'(z_k)} - \frac{1}{2}(z-z_k)L''_k(z) \end{cases}$$

Let us suppose that:

$$(5.6) \quad H_k(z) = -\int_0^z t^{\frac{2n+\alpha+\beta}{2}} (t-1)^{\frac{1-2\alpha}{2}} (t+1)^{\frac{1-2\beta}{2}} \frac{[L'_k(t) - L'_k(z_k)L_k(t)]}{(t-z_k)} dt$$

Then by using equations (5.4) and (5.5) in (5.6), we get the following equation:

$$(5.7) \quad H_k(z) = -\frac{\{n(\alpha + \beta) + (n + 1)\}}{(z_k^2 - 1)} \int_0^z t^{\frac{2n+\alpha+\beta}{2}} (t-1)^{\frac{1-2\alpha}{2}} (t+1)^{\frac{1-2\beta}{2}} L_{1k}(t) dt$$

$$+ \frac{n(\alpha - \beta)}{(z_k^2 - 1)} \int_0^z t^{\frac{2n+\alpha+\beta}{2}} (t-1)^{\frac{1-2\alpha}{2}} (t+1)^{\frac{1-2\beta}{2}} L_{1k}(t) dt$$

$$+ \frac{4z_k^2 + 2(\alpha - \beta)z_k - (\alpha + \beta + 2)(1 + z_k^2)}{(z_k^2 - 1)^2} \int_0^z t^{\frac{2n+\alpha+\beta}{2}} (t-1)^{\frac{1-2\alpha}{2}} (t+1)^{\frac{1-2\beta}{2}} L_{1k}(t) dt$$

$$+ \frac{\{2n(z_k^2 - 1) + 2\}}{z_k(z_k^2 - 1)^2} \int_0^z t^{\frac{2n+\alpha+\beta}{2}} (t-1)^{\frac{1-2\alpha}{2}} (t+1)^{\frac{1-2\beta}{2}} L_{1k}(t) dt$$

$$+ \frac{\{(\beta - \alpha)z_k - (\alpha + \beta - 1)\}}{(z_k^2 - 1)R'(z_k)} \int_0^z t^{\frac{2n+\alpha+\beta}{2}} (t-1)^{-2\alpha} (t+1)^{-2\beta} W'(t) dt$$

$$+ \frac{\{(\alpha + \beta - 2)z_k^2 + 2(\alpha - \beta)z_k + (\alpha + \beta)\}}{2z_k(z_k^2 - 1)R'(z_k)} \int_0^z t^{\frac{2n+\alpha+\beta}{2}} (t-1)^{\frac{1-2\alpha}{2}} (t+1)^{\frac{1-2\beta}{2}} W'(t) dt$$

$$- \frac{(\alpha + \beta)}{2z_k R'(z_k)} \int_0^z t^{\frac{2n+\alpha+\beta}{2}} (t-1)^{\frac{1-2\alpha}{2}} (t+1)^{\frac{1-2\beta}{2}} W'(t) dt + \frac{n}{z_k} \int_0^z t^{\frac{2n+\alpha+\beta}{2}} (t-1)^{\frac{1-2\alpha}{2}} (t+1)^{\frac{1-2\beta}{2}} \{L_k(t) - (t - z_k)L'_k(t)\} dt$$

$$- \frac{2n}{z_k(z_k^2 - 1)W'(z_k)} \int_0^z t^{\frac{2n+\alpha+\beta}{2}} (t-1)^{\frac{1-2\alpha}{2}} (t+1)^{\frac{1-2\beta}{2}} W(t) dt + \frac{1}{2} \int_0^z t^{\frac{2n+\alpha+\beta}{2}} (t-1)^{\frac{1-2\alpha}{2}} (t+1)^{\frac{1-2\beta}{2}} L_k''(t) dt$$

$$- L'_k(z_k) \int_0^z t^{\frac{2n+\alpha+\beta}{2}} (t-1)^{\frac{1-2\alpha}{2}} (t+1)^{\frac{1-2\beta}{2}} L_k(t) dt$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10} + I_{11}$$

Here, using (2.11) - (2.15), we get

$$(5.8) \quad \sum_{k=0}^{2n+1} |H_k(z)| \leq c \frac{n}{k^{-\alpha+\frac{3}{2}}}$$

Further,

$$(5.9) \quad |R'(z_k)| \sim \frac{k^{-\alpha+\frac{1}{2}}}{n^\alpha}$$

Using (5.8) and (5.9) in (4.8), we get the required lemma.

$$(5.10) \quad \sum_{k=0}^{2n+1} |A_k(z)| \leq cn^{2\alpha} \log n$$

where, $A_k(z)$ is given in (4.7) and c is a constant independent of n and z .

PROOF: We have,

$$(5.11) \quad |b_k| = |4L_k^2(z_k)| \leq cn^2$$

$$(5.12) \quad \left| z^{\frac{-2n-\alpha-\beta}{2}} (z-1)^{\frac{2\alpha-1}{2}} (z+1)^{\frac{2\beta-1}{2}} R(z) S_k \right| \leq \frac{c}{k^{2-2\alpha}}$$

Using lemma 2, lemma 3, (5.11) and (5.12) in (4.7), we get (5.10).

§6. CONVERGENCE:

Let $f(z)$ be analytic for $|z| < 1$ and continuous for $|z| \leq 1$ and $\omega(f, \delta)$ be the modulus of continuity of $f(e^{i\theta})$.

THEOREM 5: Let $f(z)$ be continuous in $|z| \leq 1$ and analytic in $|z| < 1$. Let the arbitrary number

β_k 's be such that:

$$(6.1) \quad \{\beta_k\} = o(n^2 \omega(f, n^{-1})), k = 0(1)2n+1$$

Then $\{R_n\}$ be defined by :

$$(6.2) \quad R_n(z) = \sum_{k=0}^{2n+1} f(z_k) A_k(z) + \sum_{k=0}^{2n+1} \beta_k B_k(z)$$

Satisfies the relation:

$$(6.3) \quad |R_n(z) - f(z)| = o(n^{2\alpha} \log n \omega(f, n^{-1})),$$

where, $\omega(f, n^{-1})$ is the modulus of continuity of $f(z)$.

To prove theorem 5, we shall need the following:

REMARK: Let $f(z)$ be continuous in $|z| \leq 1$ and analytic in $|z| < 1$. Then there exists a polynomial

$F_n(z)$ of degree $2n-2$ satisfying Jackson's inequality

$$(6.4) \quad |f(z) - F_n(z)| \leq cw_2(f, n^{-1}), \quad z = e^{i\theta} (0 \leq \theta < 2\pi)$$

And also an inequality due to O.Kis[3]

$$(6.5) \quad |F_n^{(m)}(z)| \leq cn^m \omega(f, n^{-1}), \quad \text{for } m \in \mathbb{I}^+$$

PROOF: Since $R_n(z)$ be given by (6.2) is a uniquely determined polynomial of degree $\leq 4n + 3$, the polynomial $F_n(z)$ satisfying (6.4) and (6.5) can be expressed as :

$$F_n(z) = \sum_{k=0}^{2n+1} F_n(z_k) A_k(z) + \sum_{k=0}^{2n+1} F_n''(z_k) B_k(z)$$

Then,

$$\begin{aligned} |R_n(z) - f(z)| &\leq |R_n(z) - F_n(z)| + |F_n(z) - f(z)| \\ &\leq \sum_{k=0}^{2n+1} |f(z_k) - F_n(z_k)| |A_k(z)| + \sum_{k=0}^{2n+1} \{|\beta_k| + |F_n''(z_k)|\} |B_k(z)| \\ &\quad + |F_n(z) - f(z)| \end{aligned}$$

Using $z = e^{i\theta} (0 < \theta \leq 2\pi)$, (6.1),(6.4),(6.5) and lemma 2 and 4, we get (6.3).

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